An Explication [1] of the Concept "Analogy"

Introduction

In this article we have tried to give an explicit interpretation to the concept "analogy", which, according to us—is used too often in an intuitive way without explicit definition.

In this interpretation we have constructed a quantification function for analogy. This quantification function is dependant on a digital automaton but under certain conditions extendible to an analog automaton. We have tried to prove that this quantitative function is a distance function.

The importance of the proof viz. that our quantification function is a distance function, seems to us firstly, that because of this, it is possible to define the surroundings of a point and from there to impose a topology to a set. Furthermore it is possible, thanks to the existence of a topology, to define continuities and limits. A topology, based on a distance function is also a hausdorffse or divided topology which entails that the limit of a row (if this limit exists) is unique ([2]).

After all this, we compared our interpretation with J. M. Bochenski's alternative-analogy-theory which he explains in this article "Ueber die Analogie" ([3]). His definition we have regarded as a specific form of analogy in our interpretation with the characteristics in respect thereto as they are given by Bochenski. We also denied—according to us—Bochenski's criticism on the alternative-analogy-concept as being trivial.

Here we would like to thank Prof. Apostel, who has strongly influenced us, also Prof. De Punt, to whose courses "Principles of differential and integral calculus, and calculus of probability", we owe a great deal and whose indications have determined the structure and the clarity of this article to a great extent. We also want to thank Dr. Crombez, who assisted

(1) By explication we understand the concept which Carnap R. defines in his "Logical Foundation of Probability" as follows: "By an explication we understand the transformation of an unexact concept, the explicandum into an exact concept the explicatum (p. 1)... The explicatum must be given by explicit rules for its use, for example by a definition, which incorporates it into a well-constructed system of scientific either logico-mathematical or empirical concepts (p. 3)... A concept must fulfil the following requirements in order to be an adequate explicatum for a given explicandum a) A simularity to the explicandum, b) Exactness, c) Fruitfulness, d) Simplicity (p. 5).

(2) For all these properties of a distance-function see Wolfgang, Franz, "Topologie", I Allgemeine Topologie, Sammlung Göschen Band 1181, Berlin 1960.

(3) Bochenski, J. M. "Logisch Philosophische Studiën," Verlag Karl Alber, Freiburg/München 1959, "Ueber die Analogie," pp. 107-129, and above all "Die Alternativtheorie," pp. 119-124.

us in the correction of the proof of the distance function. We would also like to extend our thanks to Mrs. Orlans who aided us in composing the English text.

Naturally we remain solely responsible for the eventual imperfections in our conception and its elaboration.

Description of Inputs, Abstractors, the Transmission of the Description to the Description-Model

By the description of an input we mean a definite "datum" e.g. a binary number that an input will cause the automaton to produce as one of its internal states in function of the input-unit and the main-program and in which form the input will further influence the activity of the automaton.

By description-model we mean a part of the memory-unit, in which the descriptions of the inputs are stored.

In a digital automaton A we have a number of places (receptors) along which the automaton can receive inputs. We call an input accepted when it gets a certain description which is stored in the input-register. An input-register is a composition of memory-cells (storage) which constitutes a part of the input-unit and where descriptions (for us sequences of noughts and ones) can be stored.

We can compare the receptors to the receptors of a person such as hearing, touch, smell... . The nature of the stimulus which is accepted by the input-receptors (viz. which will receive a description which will be included in the input-register) when these are ready to receive, is dependant on the characteristics of the input-cells (these are cells which form an input-receptor) and of the input-unit as a whole.

The descriptions stored in the input-register will be transmitted at the order of the main-program, after having been brought by the abstractors or not, to the description-model.

An abstractor is a part of the automaton which according to a determined principle disregards an amount of information of a description so that we can arrive at a less great diversity or even to an identity of certain descriptions ([4]). In the description-model the descriptions will be stored according to a well determined strategy ([5]). A strategy which must see that as few

(4) For elaboration of some abstractors see Culbertsen, J. T., "Consciousness and Behavior," pp. 83-126, W. M. C. Brown Company 1950.

(5) An explication of various storage-strategies may be found in our licentiate treatise "Schets voor een gedeeltelijke simulatie van het betekenisbegrip in een automaton," 1966 ("Sketch of a partial simulation of the concept of meaning in an automaton") also in Halle and Stevens "Speech Recognition," "A model and a program for research" in "The structure of language," Readings in the philosophy of Language, ed. by Jerry A. Fodor, Jerrold J. Kats, Prentice Hall Inc. New Jersy, 1964.

identical descriptions as possible will be stored in different places of the model. Then if a description appears in the input-register and if this description has been previously stored in the description-model and has not yet been erased, then by means of the storage-strategy (storage-program) this description in the model must be activated (activation e.g. by increasing the magnetism in the memory-cells in question where this description is stored).

In case the storage strategy established that the description is not yet—or no more—stored in the description model, then by order of the main-program this description will be transmitted from the input-register to a well determined place in the description-model.

The following situation is also conceivable, viz. that in the input-register descriptions are stored in order of appearance at the input receptor and once the register is full, the contents of this register is stored by means of the storage-strategy just explained.

Now one could get the impression that only descriptions from inputs are present in the model. It is however not excluded that also results of compilations with elements already previously stored in the memory are admitted, after having been brought by an abstractor or not. It is also possible that information concerning internal conditions of the automaton can be stocked. We do not exclude the possibility either, that before any information is brought in along input or internal observation, the description-model already contained description series (thus a-priori elements). These may possible be explained as necessarly or accidentally ensuing from the construction of the automaton.

With the data we have assumed up to now, solipsism remains possible, viz. in case every input would be an output of the automaton. Realism as well as idealism also remain possible.

II. Dimensions

The number of dimensions which a description will have is equal to the number of abstractors through which we must pass the description, to be left with no description at all as a final result.

This can be realized by constructing the abstractors in such a way that, that part which is taken up in a certain abstractor is substracted from a description and not that from which the abstractor will make abstraction.

We should be able to achieve this simply by the construction that an abstractor as output

a) has on the one hand that part of the description which has exercised no influence on the condition of the abstractor and

b) on the other hand the description of the internal condition of the abstractor in question and
c) —if desired—the original description.

The name of each abstractor through which the description is passed, is the name of a dimension of the description. That which is thus obsorbed by the abstractor used, forms a datum in a dimension of the description.

III. Our Interpretation of Analogy and Quantification

With an expression such as " '*a*' is analogous with '*b*' " is meant that in a certain aspect '*a*' is identical to '*b*'. Besides the problem of pointing out in which aspects '*a*' must be identical to '*b*', in such a way that '*a*' is analogous with '*b*', it is interesting to quantificate the extents to which they are analogous (the extents of being identical in the mentioned aspects).

With regards to the first problem we say in *a first approach* that '*a*' will be analogous to '*b*' when '*a*' has a number of dimensions common to '*b*' (6) or in other words, if the cross section of dimensions of '*a*' and of '*b*' is not empty. A first approach of quantification of this analogy we can obtain as follows.

Let be:

l : the number of dimensions of '*a*' which are identical with dimensions of '*b*' for the automaton X

m : the total number of dimensions of '*b*' for the automaton X

n : the total number of dimensions of '*a*' for the automaton X

K : the number of dimensions which '*a*' has and which '*b*' does not possess, increased by the number that '*b*' has and which '*a*' does not possess. Here K is always a positive number.

Then we say that '*a*' in the following extent:

$$\frac{l^2 \cdot 1}{mn \cdot (K+1)} \qquad \text{(formula I)}$$

is analogous with '*b*' for the automaton X.

A greater specification (second approach): it is obvious that different data, within one and the same abstractor, can still vary greatly. So for example within a formabstractor, an ellipse and a circle will still differ to a great extent. It will be necessary to indicate to which extent data, *within one and the same abstractor*, are analogous with one other.

(6) Bochenski in his article "Ueber die Analogie" (see note 3) also constructs an analogy on the base of "Identität einer Reiche vor formale Eigenschaften der betreffenden Beziehungen." This in his isomorphic theory.

This can be done by comparing the formulae, which express the data within a certain abstractor.

The analogy will lie in these elements, which are identical in both formulae. To quantify, we accept the rule, that the less the differences that there are between both the formulae the greater the analogy will be.

We express this by the formula

$$\frac{1}{|x|+1} \qquad \text{(formula II)}$$

$|x|$ is equal to the addition of the modulus of the differences of the components of the formula in its simplest fraction which reproduce the condition of an abstractor for this part of the description of '*b*', which is received by the abstractor in question, with the corresponding components of the formula, also in its simplest fraction, which reproduces the condition of the abstractor for this part of the description of '*a*', which is received by the abstractor in question (If the operators differ in two formulae, then x could be increased, for instance by 10, per operator). (7)

We take modulus x (the modulus of a number is the absolute value of this number) for two reasons:

a) In case a component of the formula of '*a*' is greater than the corresponding component of '*b*', than we would obtain a negative value when making the difference $b - a$. Therefore x, could possibly be negative and hence also the whole expression. To always have a positive number we take the modulus of x.

b) The second reason we take the modulus of x is, that in the formula $1/(|x|+1)$, x may never be equal to -1, because then we get zero in the dominator, owing to which we get something which is not algebraically permissable.

In the formula: $1/(|x|+1)$ if $|x| = 0$ or in other words if both the formulae are identical, we get as a result 1, and only then. As $|x|$ is larger, viz. the more the two formulae differ, the smaller the result of $1/(|x|+1)$ will become, or in other words the smaller the analogy will be, in case $1/(|x|+1)$ expresses the extent of analogy of two data in one and the same abstractor.

Within a form-abstractor, for instance we want to determine the analogy of an ellipse with a circle by means of the formula $1/(|x|+1)$.

(7) We could also construct $|x|$ as a modulus of the multiplication of the differences—between the corresponding components of the formulae—different from zero and $|x| = 0$ if no differences appear different from zero, or also $|x| =$ to the addition of the square of the differences between the corresponding elements of the formulae.

Let within this form-abstractor the formula for an ellipse be this formula, which generally is used in analytic geometry, viz.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{(see note 8)}$$

Within the same abstractor we take as formula for a circle this formula, which is also used in analytic geometry, viz.:

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1 \quad \text{(see note 8)}$$

Let in a certain ellipse $b = 5$ and $a = 2$, then the analogy between the ellipse ('a') with $b = 5$ with a circle ('b') with $a = 2$, will be dependant on the formula $1/(|x| + 1)$. Here the x is $a^2 - b^2 = 4 - 25 = -21$. $|x| = 21$ and the analogy between both is $1/(|24| + 1)$.

In the determination of the analogy of a datum 'a' with a datum 'b' *for* the automaton X, *we must take into account the extent of analogy by identical dimensions* (= abstractors), and *the extent of analogy within one and the same dimension* (abstractor).

To now construct an analogy formula, which takes both analogy quantifications into account, we can multiply this analogy, determined in function of the identical dimensions, (after having divided by the maximal number of dimensions of 'a' or 'b', which we call m) with the sum of analogies within these identical dimensions.

Finally we obtain as measure for the analogy between 'a' and 'b' for the automaton X the *formula III.*

$$\frac{l^2 \cdot 1}{mn \cdot (K+1)} \cdot \frac{1}{m} \cdot \sum_1^l \frac{1}{|x| + 1}$$

where by $\sum_1^l$ is the abbreviation for $\sum_{i=1}^l$ which indicates that the *sommation must be made of the analogies within the abstractors* 1 *to* l which are identical dimensions of the data 'a' and 'b' for the automaton X.

In case $l = m = n$, $K = 0$ and $1/(|x_i| + 1) = 1$ within each abstractor, then the formula III will be equal to 1, or in other words we then have maximal analogy of 'a' with 'b', or still: 'a' is identical with 'b' for automaton X. That formula III is here equal to 1, we see for the fact that, when $l = m = n$ and $K = 0$. We get

$$\frac{1 \cdot m^2}{m \cdot m \cdot 1} \cdot \frac{1}{m} \cdot \sum_1^m \frac{1}{|x| + 1} = \frac{1}{m} \cdot \sum_1^m \frac{1}{|x| + 1}$$

(8) Kuypers, L., and Timman, R., "Handboek der wiskunde." Deel IV. "Analytische Meetkunde," door F. Loonstra, pp. 77-84.

Because on account of the datum for each abstractor the analogy is maximal, or in other words $x = 0$, we have that in each abstractor $1/(|x| + 1) = 1$. As we here have m abstractors, we get here

$$\sum_1^m \frac{1}{|x| + 1} = m \cdot 1 = m$$

Thus

$$\frac{1}{m} \cdot \sum_1^m \frac{1}{|x| + 1} = \frac{1}{m} \cdot m = 1$$

The more the result of formula III gets smaller than 1, the more the analogy between '*a*' and '*b*' gets smaller; the more it approaches 1, the more the analogy gets larger.

IV. Our Quantification Function is a Distance Function

A distance function is a function $d: X \times X \rightarrow R^+$ by which $(a, b) \rightarrow d(a, b)$ and which satisfies the following conditions:

(1) $d(a, b) > 0$ for $a \neq b$

(2) $d(a, b) = 0$ for $a = b$

(3) $d(a, b) = d(b, a)$

(4) $d(a, b) + d(b, c) \geqslant d(a, c)$ (see note 9)

The function f_{IV}, viz. $d = (1 - (f_{(III)}))$ or

$$1 - \frac{l^2 \times 1}{m \times (K + 1)} \times \frac{1}{m} \times \sum_1^l \frac{1}{|x| + 1}$$

which we regard as the determining distance-function for analogy, satisfies the 4 above conditions.

1) *It satisfies condition* (1)

In case either $l \neq n$ or $l \neq n$ or $K \neq 0$ or $\sum_1^l 1/(|x| + 1) \neq m$ then will certainly $f_{(III)} \neq 1$ and $1 - f_{(III)} \neq 0$. In addition $f_{(III)}$ is always less than or equal to 1 and always greater than or equal to 0.

(9) De Punt, J., "Infinitesimaalrekening en beginselen der waarschijnlijkheidsrekening," unpublished course, 1964-1965, and also Patterson, E. M., "Topology," Oliver and Boyd, New York 1959, p. 26. He proves that when conditions (2) and (4) are fulfilled, then also conditions (1), (2), (3) and (4) are fulfilled.

Now, as $a \neq b$ for X, due to the construction, one of either above-mentioned inequalities will appear and $1 - f_{(III)} \neq 0$ and be more than 0.

2) *It satisfies condition* (2)

As $a = b$ for automaton X, then $f_{III} = 1$, as has already been shown above; with the result that the f_{IV}, viz. $1 - f_{(III)} = 0$.

3) *It satisfies condition* (3)

Or in other words $f_{III}(a, b) = f_{III}(b, a)$.

If 'a' possesses (n) dimensions and 'b' ($n + 2$) dimensions and in addition 'a' and 'b' have $n - y$ dimensions in common, we obtain for the part $(l^2 \times 1)/[n \cdot m \times (K + 1)]$, for $f_{III}(a, b)$:

$l = n - y$
$m = n + z$
$n = n$
$K = 2y + z$ ('a' possesses y dimensions which 'b' does not possess
'b' possesses $y + z$ dimensions which 'a' does not possess)

and hence

$$\frac{(n - y)^2 \times 1}{(n)(n + z) \times (2y + z + 1)}$$

For this same part we get for $f_{III}(b, a)$

$$\begin{aligned} l &= n - y \\ m &= n \\ n &= n + z \\ K &= 2y + z \end{aligned}$$

and hence:

$$\frac{(n - y)^2 \times 1}{(n + z)(n) \times (2y + z + 1)}$$

It is clear that for this part for $f_{III}(a, b)$ and $f_{III}(b, a)$ we get an identical result.

Due to the construction we must multiply these identical parts in both cases with $1/m$; m in both cases is $= (n + z)$ or the greatest number of dimensions which is to be found in those compared.

For the second part of the formula for $f_{III}(a, b)$ and $f_{III}(b, a)$ the summation $\sum_1^l$ will be equal for both, as the l is the same for both, in case within each abstractor the analogy of (a, b) is still now equal.

The analogy within the abstractor is determined by the formula II $1/(|x| + 1)$. In this formula the only variable is x. $|x|$ is due to construction (see above) the modulus of the result of the addition of the modulus of the differences between both the formulae, which reproduces the condition of the abstractor (which is a dimension of 'a' and 'b') when 'a' is present and when 'b' is present. The differences between the formulae for (a, b) or (b, a) will have one and the same value. And the addition of these absolute values will also be identical for (a, b) as for (b, a).

Thus the condition (3) is also satisfied, because when $f_{III}(a, b) = f_{III}(b, a)$ then is $f_{IV}(a, b) = f_{IV}(b, a)$.

4) *The function f_{IV} satisfies condition* (4)

As reminder:

$$(f_I) = \frac{l^2 \times 1}{nm \times (K+1)}$$

$$(f_{II}) = \frac{1}{|x| + 1}$$

$$(f_{III}) = \frac{l^2 \times 1}{nm \times (K+1)} \times \frac{1}{m} \times \sum_1^l \frac{1}{|x|+1}$$

$$(f_{IV}) = (1 - f_{III})$$

We regard the following data:

Let in A $n = r$ (n is the number of dimensions)
Let in B $n = s$ and $r \geqslant s \geqslant v$
Let in C $n = v$

hence:

$$l_{ab} = \begin{matrix} t_1 < r \\ t_1 \leqslant s \end{matrix} \qquad K_{ab} = (r - t_1) + (s - t_1) \qquad x_{ab} \in \{Vt_1\}$$

$$l_{bc} = \begin{matrix} t_2 < s \\ t_2 \leqslant v \end{matrix} \qquad K_{bc} = (s - t_2) + (v - t_2) \qquad x_{bc} \in \{Vt_2\}$$

$$l_{ac} = \begin{matrix} t_3 < r \\ t_3 \leqslant v \end{matrix} \qquad K_{ac} = (r - t_3) + (v - t_3) \qquad x_{ac} \in \{Vt_3\}$$

With these data we can construct the following functions:

$$f_{III}(a, b) = \frac{(t_1)^2 \times 1}{rs \times [(r - t_1) + (s - t_1) + 1]} \times \frac{1}{r} \times \sum_1^{t_1} \frac{1}{|x_{ab}| + 1}$$

$$f_{III}(b, c) = \frac{(t_2)^2 \times 1}{sv \times [(s - t_2) + (v - t_2) + 1]} \times \frac{1}{s} \times \sum_1^{t_2} \frac{1}{|x_{bc}| + 1}$$

$$f_{III}(a, c) = \frac{(t_3)^2 \times 1}{rv \times (r - t_3) + (v - t_3) + 1} \times \frac{1}{r} \times \sum_1^{t_3} \frac{1}{|x_{ac}| + 1}$$

It remains to be proved

$$
\begin{aligned}
(1 - f_{III}(a, b)) + (1 - f_{III}(b, c)) &\geqslant (1 - f_{III}(a, c)) \\
(2 - f_{III}(a, b)) + f_{III}(b, c) &\geqslant 1 - f_{III}(a, c) \\
1 - (f_{III}(a, b) + f_{III}(b, c)) &\geqslant -f_{III}(a, c) \\
-f_{III}(a, c) &\leqslant 1 - (f_{III}(a, b) + f_{III}(b, c))
\end{aligned}
$$

$$
\boxed{-1 - (f_{III}(a, c)) \leqslant -(f_{III}(a, b) + f_{III}(b, c))} \tag{I}
$$

The method we will follow to prove the inequality (I) is the following:

A: $r > s$ or $s > v$.

The following cases remain possible here

A.a: $r > s > v$.

The first member of the inequality (I) is always less than -1 (argumentation: see further). If every part of the addition from the second member $\leqslant \frac{1}{2}$ then its negation will always be $\geqslant -1$ and as a result the inequality will be proved.

A.b. when $r = s > v$, then the inequality remains valid.

A.c: when $r > s = v$ then idem.

B: What about the inequality (I) if $r = s = v$? The first question which presents itself here is which possibilities remain. These are:

B.a: $t_1 = t_2 = t_3 = r$

$$\sum_{1}^{t_1} \frac{1}{|x_{ab}| + 1} = r$$

$$\sum_{1}^{t_2} \frac{1}{|x_{bc}| + 1} = r$$

$$\sum_{1}^{t_3} \frac{1}{|x_{ac}| + 1} = r$$

and thus $a = b = c$

B.b: $t_1 < r$ and $t_2 < r$. Here the same reasoning as in *A*. Independant of the value of t_3, the inequality will be proved here.

B.c: $t_1 = r$ or $t_2 = r$ or both $= r$.

B.c.1: in case $t_1 = t_2 = r$.

B.c.2: $t_1 = r$ and $t_1 \neq t_2$ or reversed, in both cases the proof is to be made in the same way.

C: That $f_{IV}(ab) + f_{IV}(bc) \geqslant f_{IV}(ac)$ will be proved by A and B. Now there still remains to be proved that $f_{IV}(bc) + f_{IV}(ac) \geqslant f_{IV}(ab)$ and $f_{IV}(ab) + f_{IV}(ac) \geqslant f_{IV}(bc)$. These two inequalities may be proved in the same way as $f_{IV}(ab) + f_{IV}(bc) \geqslant f_{IV}(ac)$.

Proof that in above- mentioned possible cases inequality is valid.

A.a: The first part of the inequality (I) (we call this *Ia*) is always less than -1, in case $f_{III}(a, c)$ is always positive

$$f_{III}(a, c) = \frac{(t_3)^2 \times 1}{rv \times [(r - t_3) + (r - t_3) + 1]} \times \frac{1}{r} \times \sum_1^{t_3} \frac{1}{x_{ac} + 1}$$

Due to construction $t, r, v, \mid x_{ac} \mid$ is always positive. Due to construction $t_3 \leqslant r$ and $t_3 \leqslant v$ is also always positive.

As a result we have that

$$r - t_3 \geqslant 0$$
$$v - t_3 \geqslant 0$$

Therefor the first member of the inequality (*Ia*) is always less than or equal to -1. The second member of the inequality I (we call this member *Ib*) consists of the negative of an addition. In case each term of this addition is less than $\frac{1}{2}$, its negative will always be greater than -1, and as a result the second member of the inequality will always be greater than the first member (what has to be proved). We shall thus look under which conditions $f_{III}(a, b) \leqslant \frac{1}{2}$ and $f_{III}(a, b) \leqslant \frac{1}{2}$, and then examine if in the cases where $f_{III}(a, b) > \frac{1}{2}$ or $f_{III}(b, c) > \frac{1}{2}$ or both $> \frac{1}{2}$ the inequality is still valid.

$$f_{III}(a, b) = \frac{(t_1)^2 \times 1}{rs \times [(r - t_1) + (s - t_1) + 1]} \times \frac{1}{r} \times \sum_1^{t_1} \frac{1}{\mid x_{ab} \mid + 1}$$

As we must prove that $f_{III}(a, b) \leqslant \frac{1}{2}$, then, if we take the maximum value of $f_{III}(a, b)$ and if this maximum value still $\leqslant \frac{1}{2}$, will be proved that $f_{III}(a, b) \leqslant \frac{1}{2}$.

The max. value of

$$\sum_1^{t_1} \frac{1}{\mid x_{ab} \mid + 1} = t_1 \quad \text{(see above)}$$

$$\frac{(t_1)^2 \times 1}{rs \times [(r - t_1) \times (s - t_1) + 1]} \times \frac{1}{r} \times t_1 \leqslant \frac{1}{2}$$

The maximal $t_1 = s$ (datum)

$$\frac{s^2 \times 1}{rs \times (r - s + s - s + 1)} \times - \frac{s}{r} \leqslant \frac{1}{2}$$

$s \leqslant r$ due to datum

$$\frac{s^2}{rs \times (r - s) + (s - s) + 1} \leqslant \frac{1}{2}$$

$$\frac{s}{r^2 - rs + r} \leqslant \frac{1}{2}$$

$$2s \leqslant r^2 - rs + r$$

$$(r + 2)s \leqslant r(r + 1)$$

$$s \leqslant \frac{r(r + 1)}{r + 2} < r$$

$s < r$. Thus if $\boxed{s < r}$ the inequality is true. When $r = s$ then as opposed to it, the inequality is not satisfied.

For $f_{III}(b, c) \leqslant \frac{1}{2}$ we have an analogous proof.

Conclusion: From the proof of (*Ia*) and (*Ib*) it follows that in case $r > s > v$ the triangular-inequality $f_{IV}(a, b) + f_{IV}(b, c) \geqslant f_{IV}(a, c)$ applies.

B: We must now examine what happens when $r = s = v$.

B(a): If $a = b = c$ then $r = s = v$ and $t_1 = t_2 = t_3$

$$\sum_{1}^{t_1=r} \frac{1}{|x_{ab}| + 1} = r$$

$$\sum_{1}^{t_2=r} \frac{1}{|x_{bc}| + 1} = r$$

$$\sum_{1}^{t_3=1} \frac{1}{|x_{ac}| + 1} = r$$

and $f_{III}(a, c) = 1$; $f_{III}(a, b) = 1$; $f_{III}(b, c) = 1$ (reason see above).

The inequality I will then be: $-1 - 1 \leqslant -(1 + 1)$ and will thus be satisfied.

B(b): Another possibility is that $a \neq b \neq c$ but $r = s = v$, then it is possible that $t_1 \neq t_2 \neq t_3$ or

$$\sum_{1}^{t_1} \frac{1}{|x_{ab}| + 1} \neq \sum_{1}^{t_2} \frac{1}{|x_{bc}| + 1} \neq \sum_{1}^{t_3} \frac{1}{|x_{ac}| + 1}$$

The first member of the inequality (I), with these facts, will also always be less than -1, because $f_{III}(a, c)$ is always positive (see above).

Here also in case each member of the addition of the second member of the inequality is less than $\frac{1}{2}$, the inequality (I) will be valid. This is so in case $t_1 < r$ and $t_2 < r$.

$f_{III}(a, b) < \frac{1}{2}$:

$$f_{III}(a, b) = \frac{(t_1)^2 \times 1}{r^2(2r - 2t_1 + 1)} \times \frac{1}{r} \times \sum_1^{t_1} \frac{1}{|x_{ab}| + 1}$$

$$\leqslant \frac{(t_1)^2 \times t_1}{r^2(2r - 2t_1 + 1) \times r}$$

$$\leqslant \frac{(t_1)^2}{r^2(2r - 2t_1 + 1)} \qquad t_1 \leqslant r \quad \text{according to data}$$

$$\leqslant \frac{(t_1)^2}{r^2[2(r - t_1) + 1]}$$

In the event that we have $t_1 < r$, and as r and t_1 represent numbers from the row of natural numbers, as a result the minimum $r - t_1 = 1$.

Hence:

$$\leqslant \frac{(t_1)^2}{r^2(2 + 1)} < \frac{1}{2 + 1} \leqslant \boxed{\frac{1}{3} < \frac{1}{2}}$$

The second part of the addition from the second member of the inequality (I) with the data from $B(b)$ is proved in the same way. Thus if $t_2 < r$ then the inequality (I) is also proved.

B(c): The condition only remains where:

$$a \neq b \neq c \qquad r = s = v$$

$$t_1 = r \quad \text{or} \quad t_2 = r \quad \text{or} \quad (t_1 \text{ and } t_2 = r).$$

Here we can distinguish two possible cases:

B(c.1): In case t_1 and t_2 are equal to r then t_3 will also be equal to r and '*a*' and '*b*' and '*c*' will have the same dimensions.

B(c.2): In case $t_1 = r$ and $t_1 \neq t_2$ then t_2 will be equal to t_3 and '*a*' and '*b*' will have the same dimensions.

The Case B(c.3): $t_2 = r$ and $t_1 = t_2$ then $t_1 = t_3$ and '*b*' and '*c*' will have the same dimensions. The proof of this happens in the same way as B(c.2).

$B(c.1)$: $a \neq b \neq c$; $r = s = v$; $t_1 = t_2 = t_3 = r$. Here the difference will lie in

$$\sum_1^{t_1} \frac{1}{|x_{ab}|+1} \neq \text{or} = \sum_1^{t_2} \frac{1}{|x_{bc}|+1} \neq \text{or} = \sum_1^{t_3} \frac{1}{|x_{ac}|+1}$$

Here we again get inequality (I). Every part of the inequality can be divided by:

$$\frac{r^2 \times 1}{r^2 \times [(r-r)+(r-r)+1)} \times \frac{1}{2}$$

as this expression belongs to the $f_{III}(a, b)$, $f_{III}(b, c)$, $f_{III}(a, c)$. We obtain the result.

$$\frac{-1 \times r^2 \times (r-r+r-r+1) \times r}{r^2 \times 1} - \sum_1^r \frac{1}{|x_{ac}|+1}$$

$$\leqslant \left(-\sum_1^r \frac{1}{|x_{ab}|+1} + \sum_{bc}^r \frac{1}{|x_{bc}|+1}\right)$$

or simplified

$$\boxed{-r - \sum_1^r \frac{1}{|x_{ac}|+1} \leqslant -\left(\sum_1^r \frac{1}{|x_{ab}|+1} + \sum_1^r \frac{1}{|x_{bc}|+1}\right)}$$

Now:

a) $$\sum_1^r \frac{1}{|x_{ab}|+1} = \sum_1^{r-k} \frac{1}{|x_{ab}|+1} + \sum_1^k \frac{1}{|x_{ab}|+1}$$

k is the number of dimensions where $|x_{ab}| \neq 0$. *where* $|x_{ab}| = 0$ *will* $|x_{bc}| = |x_{ac}|$

b) $$\sum_1^r \frac{1}{|x_{ac}|+1} = \sum_1^{r-k} \frac{1}{|x_{ac}|+1} + \sum_1^k \frac{1}{|x_{ac}|+1}$$

c) $$\sum_1^r \frac{1}{|x_{bc}|+1} = \sum_1^{r-k} \frac{1}{|x_{bc}|+1} + \sum_1^k \frac{1}{|x_{bc}|+1}$$

$$= \sum_1^{r-k} \frac{1}{|x_{ac}|+1} + \sum_1^k \frac{1}{|x_{bc}|+1}$$

Thus:

$$-r-\sum_1^{r-k}\frac{1}{|x_{ac}|+1}-\sum_1^{k}\frac{1}{|x_{ac}|+1}\leqslant -\sum_1^{r-k}\frac{1}{|x_{ab}|+1}-\sum_1^{k}\frac{1}{|x_{ab}|+1}$$

$$-\sum_1^{r-k}\frac{1}{|x_{ac}|+1}-\sum_1^{k}\frac{1}{|x_{bc}|+1}-r-\sum_1^{k}\frac{1}{|x_{ac}|+1}\leqslant -\sum_1^{r-k}\frac{1}{|x_{ab}|+1}$$

$$-\sum_1^{k}\frac{1}{|x_{ab}|+1}-\sum_1^{k}\frac{1}{|x_{bc}|+1}\quad\sum_1^{r-k}\frac{1}{|x_{ab}|+1}=r-k$$

$$-r-\sum_1^{k}\frac{1}{|x_{ac}|+1}\leqslant -r+k-\sum_1^{k}\frac{1}{|x_{ab}|+1}-\sum_1^{k}\frac{1}{|x_{bc}|+1}$$

$$-k-\sum_1^{k}\frac{1}{|x_{ac}|+1}\leqslant -\sum_1^{k}\frac{1}{|x_{ab}|+1}-\sum_1^{k}\frac{1}{|x_{bc}|+1}$$

(The last inequality we call the inequality III)

$$k+\sum_1^{k}\frac{1}{|x_{ac}|+1}\geqslant +\sum_1^{k}\frac{1}{|x_{ab}|+1}+\sum_1^{k}\frac{1}{|x_{bc}|+1}$$

k (a natural number) is the number of dimensions where $|x_{ab}|\neq 0$; every element from the sommation is thus smaller or equal to $\frac{1}{2}$ (x_i is always a whole number see earlier).

Thus:

$$\sum_1^{k}\frac{1}{|x_{ab}|+1}\leqslant\frac{k}{2}$$

or generally

$$\sum_1^{p}\frac{1}{|x_{ab}|+1}<\frac{p}{2}$$

where $1\leqslant P\leqslant k$

$$\sum_1^{k}\frac{1}{|x_{bc}|+1}=\sum_1^{k-l}\frac{1}{|x_{bc}|+1}+\sum_1^{l}\frac{1}{|x_{bc}|+1}$$

l is the number of dimensions from the k dimensions where $|x_{bc}|\neq 0$

$$\sum_1^{k}\frac{1}{|x_{ab}|+1}=\sum_1^{k-l}\frac{1}{|x_{ab}|+1}+\sum_1^{l}\frac{1}{|x_{ab}|+1}$$

$$\sum_1^{k}\frac{1}{|x_{ac}|+1}=\sum_1^{k-l}\frac{1}{|x_{ac}|+1}+\sum_1^{l}\frac{1}{|x_{ac}|+1}$$

As in the dimensions $k - l$, $|x_{bc}| = 0$ shall $|x_{ac}| = |x_{ab}|$ and thus:

$$\sum_{1}^{k-l} \frac{1}{|x_{ac}|+1} = \sum_{1}^{k-l} \frac{1}{|x_{ab}|+1}$$

$$k + \sum_{1}^{k-l} \frac{1}{|x_{ac}|+1} + \sum_{1}^{l} \frac{1}{|x_{ac}|+1} \geqslant \sum_{1}^{k-l} \frac{1}{|x_{ab}|+1} + \sum_{1}^{l} \frac{1}{|x_{ab}|+1}$$

$$+ \sum_{1}^{k-l} \frac{1}{|x_{bc}|+1} + \sum_{1}^{l} \frac{1}{|x_{bc}|+1}$$

$$k + \sum_{1}^{l} \frac{1}{|x_{ac}|+1} \geqslant \sum_{1}^{l} \frac{1}{|x_{ab}|+1} + \sum_{1}^{k-l} \frac{1}{|x_{bc}|+1}$$

$$+ \sum_{1}^{l} \frac{1}{|x_{bc}|+1} \quad \sum_{1}^{k-l} \frac{1}{|x_{bc}|+1} = k - l$$

$$l + \sum_{1}^{l} \frac{1}{|x_{ac}|+1} \geqslant \sum_{1}^{l} \frac{1}{|x_{ab}|+1} + \sum_{1}^{l} \frac{1}{|x_{bc}|+1}$$

Then as

$$\sum_{1}^{l} \frac{1}{|x_{ab}|+1} \leqslant \frac{l}{2} \quad \text{and} \quad \sum_{1}^{l} \frac{1}{|x_{bc}|+1} \leqslant \frac{l}{2}$$

the inequality is proved.

$B(c.2)$: $a \neq b \neq c \qquad r = s = v$

$$t_1 \neq t_2$$
$$t_1 \neq t_3$$
$$t_1 = r$$

if $t_1 \neq t_2$ and $t_1 \neq t_3$ then $t_2 = t_3$

$$-1 - \frac{(t_3)^2 \times 1}{r^2[(r - t_3 + r - t_3) + 1]} \times \frac{1}{r} \times \sum_{1}^{t_3} \frac{1}{|x_{ac}|+1}$$

$$\leqslant -\left(\frac{r^2 \times 1}{(r^2 \times 1)r} \times \sum_{1}^{r} \frac{1}{|x_{ab}|+1} + \frac{(t_3)^2 \times 1}{r^2(r - t_3 + r - t_3 + 1)}\right.$$

$$\left.\times \frac{1}{r} \times \sum_{1}^{t_3} \frac{1}{|x_{bc}|+1}\right)$$

$$\leqslant -\left(\frac{1}{r} \times \sum_{1}^{r} \frac{1}{|x_{ab}|+1} + \frac{(t_3)^2}{r^2(2r - 2t_3 + 1)} \times \frac{1}{r} \times \sum_{1}^{t_3} \frac{1}{|x_{bc}|+1}\right)$$

$$-1 \leqslant -\frac{1}{r}\sum_{1}^{r}\frac{1}{|x_{ab}|+1} - \frac{(t_3)^2}{r^2(2r-2t_3+1)} \times \frac{1}{r}\left(\sum_{1}^{t_3}\frac{1}{|x_{bc}|+1} - \sum_{1}^{t_3}\frac{1}{|x_{ac}|+1}\right)$$

$$-r \leqslant \sum_{1}^{r}\frac{1}{|x_{ab}|+1} - \frac{(t_3)^2}{r^2(2r-2t_3+1)}\left(\sum_{1}^{t_3}\frac{1}{|x_{bc}|+1} - \sum_{1}^{t_3}\frac{1}{|x_{ac}|+1}\right)$$

$$-r \leqslant -\sum_{1}^{r}\frac{1}{|x_{ab}|+1} - \frac{(t_3)^2}{r^2(2r-2t_3+1)}\left(\sum_{1}^{k}\frac{1}{|x_{bc}|+1} - \sum_{1}^{k}\frac{1}{|x_{ac}|+1}\right)$$

$$r - \sum_{1}^{r}\frac{1}{|x_{ab}|+1} \geqslant \frac{(t_3)^2}{r^2(2r-2t_3+1)}\left(\sum_{1}^{k}\frac{1}{|x_{bc}|+1} - \sum_{1}^{k}\frac{1}{|x_{ac}|+1}\right)$$

r is the number of dimensions in A.
$r - (k + b')$ is the number of dimensions where $x_{ab} = 0$.
k is the number of dimensions where $x_{ab} \neq 0$ and which belong to C.
b' is the number of dimensions which in $x_{ab} \neq 0$ but which don't belong to c.

$$r - (k + b') + (k + b') = r$$

$$r - \left(\sum_{1}^{r-(k+b')}\frac{1}{|x_{ab}|+1} + \sum_{1}^{k}\frac{1}{|x_{ab}|+1} + \sum_{1}^{b'}\frac{1}{|x_{ab}|+1}\right)$$

$$\geqslant \sum_{1}^{k}\frac{1}{|x_{bc}|+1} - \sum_{1}^{k}\frac{1}{|x_{ac}|+1}$$

$$r - \sum_{1}^{r-(k+b')}\frac{1}{|x_{ab}|+1} - \sum_{1}^{k}\frac{1}{|x_{ab}|+1} - \sum_{1}^{b'}\frac{1}{|x_{ab}|+1}$$

$$\geqslant \sum_{1}^{k}\frac{1}{|x_{bc}|+1} - \sum_{1}^{k}\frac{1}{|x_{ac}|+1}$$

$$r - (r - (k + b')) - \sum_{1}^{k}\frac{1}{|x_{ab}|+1} - \sum_{1}^{b'}\frac{1}{|x_{ab}|+1}$$

$$\geqslant \sum_{1}^{k}\frac{1}{|x_{bc}|+1} - \sum_{1}^{k}\frac{1}{|x_{ac}|+1}$$

$$r - r + k + b' - \sum_{1}^{k}\frac{1}{|x_{ab}|+1} - \sum_{1}^{b'}\frac{1}{|x_{ab}|+1}$$

$$\geqslant \sum_{1}^{k}\frac{1}{|x_{bc}|+1} - \sum_{1}^{k}\frac{1}{|x_{ac}|+1}$$

$$k + b' - \sum_1^{b'} \frac{1}{|x_{ab}| + 1} \geqslant \sum_1^{k} \frac{1}{|x_{bc}| + 1} - \sum_1^{k} \frac{1}{|x_{ac}| + 1} + \sum_1^{k} \frac{1}{|x_{ab}| + 1}$$

$$k + b' + \sum_1^{k} \frac{1}{|x_{ac}| + 1} \geqslant \sum_1^{k} \frac{1}{|x_{bc}| + 1} + \sum_1^{k} \frac{1}{|x_{ab}| + 1} + \sum_1^{b'} \frac{1}{|x_{ab}| + 1}$$

$$\sum_1^{k} \frac{1}{|x_{ab}| + 1} \leqslant k$$

$$\sum_1^{b'} \frac{1}{|x_{ab}| + 1} \leqslant b'$$

We take the maximal value of

$$\sum_1^{b'} \frac{1}{|x_{ab}| + 1} = b',$$

then we get the inequality.

$$k + \sum_1^{k} \frac{1}{|x_{ac}| + 1} \geqslant \sum_1^{k} \frac{1}{|x_{bc}| + 1} + \sum_1^{k} \frac{1}{|x_{ab}| + 1} \quad \text{(inequality (IV)}$$

Proof of this inequality is identical to the proof of the inequality III in B.c.1.

So also the case B.c.1 satisfies the triangular inequality

A.b:

$r = s > v$

if $r = s$ then $f_{\mathrm{III}}(a, b) > \frac{1}{2}$ if $t_1 = r$ and $\sum_1^{t_1} \geqslant \frac{t_1}{2} = \frac{r}{2}$

As $s \neq v$ will $f_{\mathrm{III}}(b, c) < \frac{1}{2}$

if $t_1 = r$ then will $t_3 = t_2$

When working out the inequality (I), taking into account the above data we get:

$$-1 - \left(\frac{(t_3)^2}{rv \times ((r - t_3) + (v - t_3) + 1)} \times \frac{1}{r} \times \sum_1^{t_3} \frac{1}{|x_{ac}| + 1}\right)$$

$$\leqslant -\left(\frac{1}{r} \times \sum_1^{t_1} \frac{1}{|x_{ab}| + 1} + \frac{(t_3)^2}{sr \times (r - t_3 + v - t_3 + 1)} \times \frac{1}{r} \times \sum_1^{t_3} \frac{1}{|x_{bc}| + 1}\right)$$

Proof happens in the same way as that of B.c.2.

A.c: $r > s = v$ if $s = v$ then $f_{\mathrm{III}}(b, c) > \frac{1}{2}$ if $t_2 = s = v$ (see B.b) and

$$\sum_{1}^{t_2} \frac{1}{|x_{bc}| + 1} \geqslant \frac{t_2}{2} = \frac{s}{2}$$

if $t_2 = s$ then $t_1 = t_3$.

When working out the inequality (I) taking into account the above data we get:

$$1 - \frac{(t_3)^2 \times 1}{rs \times (r - t_3 + s - t_3 + 1)} \times \frac{1}{r} \times \sum_{1}^{t_3} \frac{1}{|x_{ac}| + 1}$$

$$\leqslant -\left(\frac{(t_3)^2}{rs \times (r - t_3) + (s - t_3) + 1} \times \frac{1}{r} \times \sum_{1}^{t_3} \frac{1}{|x_{ab}| + 1} + \frac{1}{s} \times \sum_{1}^{t_2 = s} \frac{1}{|x_{bc}| + 1}\right)$$

It is proved in the same way as B.c.2.

Conclusion from A, B, and C follows that the function f_{IV} satisfies the conditions (4).

Thus as the function f_{IV} also satisfies the conditions (1) and (2) and (3) f_{IV} is a distance-function.

V. **The Alternative-Analogy-Theory as Expounded by Bochenski** (see note 3)

Important concepts are:

a) Begriff der Bedeutung: "der Name a bedeutet in der sprache l den Gehalt f am Ding x" (symbolisch: "$S(a, l, f, x)$" (Denotatum concept: the name 'a' denotes in the language l the ratio f for the object x. Symbolic:" $S(a, l, f, x)$).

b) "Die Definition für die Mehrdeutigkeit seht, wie folgt aus:

$$Ae(a, b, l, f, g, x, y) = Df \cdot S(a, l, f, x) \cdot S(b, l, g, y) \cdot I(a, b) \cdot (x \neq y)(f \neq g)$$

(The definition for one-many relations is as follows:

$$Ae(a, b, l, f, g, x, y) = \ldots).$$

$I(a, b)$ symbolisch vor a und b die gleiche graphische Gestalt haben. ($I(a, b)$ symbolic for 'a' and 'b' which have identical shapes.)

c) Definition of alternative analogy.

$$Anp(a, b, l, f, g, x, y) = Dd \cdot Ae(a, b, l, f, g, x, y) \cdot [(\exists h) \cdot (f = g \cup h)]$$
$$\text{where } [g \cup h] \cdot x = Df \cdot gx \vee hx$$

VI. Our Interpretation of the Alternative-Theory as it is expounded by Bochenski, Taking I-IV into Account

A part of the descriptions of inputs (see I) will however, when they are transmitted to the description-model, not only activate their own description through the storage-program. Through medium of the description-model-program ([10]) it is possible that, when a certain part of the description-model is activated (this part we call the *activating*) another part of the model can also be activated (the *activated*). The inputs, of which the respective descriptions possesse this property of thus not only activating these cells in the model-description in which the respective descriptions are stored, but also other cells from the description-model, will be a proper subclass of the inputs. This class we call I_t. The class with the respective descriptions of the inputs which belong to I_t we call B_t.

Let 'a' be an input belonging to I_t, let 'f' be the description which is activated in the description-model, which is different from the description of 'a' in the description-model. Let x be the input which gets as description 'f', let l be the description language of the automaton.

Then we get a certain interpretation for $S(a, l, f, x)$. 'a' will be a sign (see note 9) for the data x with the description 'f' (description relative to the automaton).

As definition for analogy we then state:

$$Anp_1(a, b, l, f, g, x, y) = Df \cdot Ae(a, b, l, f, g, x, y).$$

$$[(\exists h) \cdot f = ((f \cap g) \cup h) \cdot (f \cap g) \neq L] \quad (L = \text{null class})$$

A possible quantification for analogy is possible here by determing the extent that the $g \subset f$ or in other words the size of $f \cap g$, and thereby taking into account the extent of h.

For this quantification our quantification function can be used (see sections III).

Bochenski's alternative analogy definition forms a special place in our analogy interpretation viz. the case of $f \cap g = g$. In this case the proof that Bochenski quotes in his "Ueber die Analogie" (see note 3) for the soundness of the syllogism with analogous middle terms is also valid.

(10) For more about this approach of the concepts of sign, denotatum, meaning see our licentiate treatise "Schets voor een gedeeltelijke simulatie van het betekenisbegrip in een automaton" ("Sketch of a partial simulation of the concept of meaning in an automaton"), and also our articles: "Esquisse d'un essai de simulation de language dans un automate en vue d'éclairer la notion 'signification'," from "Actes du XIIIe congrès des sociétés de philosophie de la langue française" te verschijnen in augustus 1966, and "Sketch of a partial simulation of the concept meaning in an automaton" to appear in "Logique et Analyse."

VII. Comments on the Expounding of the Alternative Analogy Theory by Bochenski

In his criticism of the alternative-analogy-theory Bochenski says: "In Folge dieser Definition wäre *jedes Paar von Namen analog*, denn wir können immer einen neuen Namen in unser System einführen der nach Definition die Besagte Alternative bedeutet." ("As a result of this definition *every pair of names is analogous* because we can always include a new name in our system which according to the definition indicates the alternative in question".)

We agree with Bochenski that for every g, always an 'h' or an 'f' can be constructed in such a way that $f = g \cup h$. This involves that for every name, another name can be constructed so that there will be analogy between the two names. However, we do not agree that between no matter which two names an analogy will exist in his alternative-analogy-interpretation, as 'a' will only be analogous with 'b' (let f be the contents of 'a' and 'g' the contents of 'b') when $g \subset f$ ($f = g \cup h$ by analogy). And not with no matter which two names will the g be included in the f. Besides to have analogy in this definition the one-many-relation requirements must also be fulfilled and thus, amongst others, "Isomorphy" must exist between the two names, viz. $[I(a, b)]$.

Another consideration which we would like to make for Bochenski's-analogy-theory is that we consider—if we disregard historical considerations—the demand for $I(a, b)$ rather arbitrary for analogy. Let us assume $\sim I(a, c)$. Let $S(a, l, f, x)$ and let $S(c, l, g, y)$ and let the other conditions for analogy be fulfilled (amongst others $f = g \cup h$) then 'a' will not be analogous with 'c'. If we now however define 'd' as $S(d, l, g, y)$ and if we form 'd' in such a way that $I(a, d)$ is fulfilled, then there will be analogy between 'a' and 'd'.

In both cases however the syllogism with the respective middle-terms will remain valid.

F. VANDAMME

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