Entailment

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One of the problems which seems to be with us forever is about the status of the so-called paradoxes of (strict) implication. The problem arises because in most systems of formal logic, theorems of the following form are derivable:

(1) $(p \wedge \sim p) \rightarrow q$

(2) $q \rightarrow . p \vee \sim p$

inviting some logicians to say the following are principles of logic: that from the logically impossible every proposition follows; and that the logically necessary follows from every proposition. Indeed whereas $p \vee \sim p$ is a theorem in ordinary systems of formal propositional logic, $p \wedge \sim p$ is the negation of a theorem; the former entities receiving the predicate 'logically true', the latter the predicate 'logically false'.

Some logicians would not deny that the said principles are valid. But others, notably Anderson and Belnap would urge to 'take the word 'from' in 'follows from' seriously', and argue that relevance is a necessary condition for an entailment (logical consequence) to hold: if an entailment holds, the antecedent must be relevant for the consequent [1]. I shall have to say more about this later on.

Why should one be worried about a satisfactory theory of entailment? The answer, I think, is that entailment is itself the answer to the question what it is that justifies our inferring the proposition that P from the proposition that Q. It should be noted that I am interested in propositions as the relata of an entailment. Why propositions and not sentences? Let me only say this, that it is because of mistaking sentences for propositions that entailment-paradoxes seem to have become a topic philosophers care to write about. I hope to make this clear in the pages that follow.

There have been different attempts at explaining the paradoxes away. But to my knowledge most of them simply replace entailment with another relation as the fundamental relation of logic. Most prominent is the, what I shall call 'semantic' approach to logic. The approach was set out in detail by A. Tarski. It can be summarized as follows:

'Let L be any class of sentences. We replace all extralogical constants which occur in the sentences belonging to L by corresponding variables, like constants being replaced by like variables, and unlike by unlike. In this way we obtain a class L' of sentential functions. An arbitrary sequence of objects which satisfies every sentential function of the class L' will be called a *model* or *realization* of the class L of sentences (...). If in particular the class L consists of a single sentence X, we shall also call the model of the class L the model of the sentence X. In terms of these concepts we can define the concept of logical consequence as follows:
The sentence X follows logically *from the sentences of the class K if, and only if, every model of the class K is also a model of the sentence X.*' [6]

Although Tarski undoubtedly intended to defend the view that the relation of logical consequence holds between statements in virtue of their 'logical' form, it must be admitted that the basic notions of his account are to be classified as belonging to semantics. Indeed we can only know whether a sequence of objects satisfies a sentential function when we are allowed to look at what is the case, to look for examples and counter-examples. Of course, the method is 'safe' in that we can never go from a true premiss to a false conclusion. But this could hardly be accepted as the justification of an inference, although every justified inference must be safe. The semantical approach is also very prominent in the construction of models for modal logic. Here the aim is interpretation. Thus e.g. 'p entails q' is interpreted as 'in every possible world in which p is a true position, q is also a true position.' It is at once obvious that this way of talking removes all that is paradoxical about the so-called paradoxes of implication. That the approach is not satisfactory in se- i.e. apart from the fact that is may help to suggest 'interpretations' for formal calculi, and thus make the calculi easier to study – is rapidly shown. To take but the very popular Kripke-styled semantics; even in the strongest of these systems (S5-Models) one cannot prove the validity of the obvious logical truth, that it is not logically necessary that there are two different objects: $\sim L (\exists x P x \wedge \exists x \sim P x)$ is not provable [5]. So it seems hardly likely that a deficient concept of logical neccesity could yield a satisfactory concept of entailment- which is defined as 'it is logically necessary that ... materially implies...'. Granting, that is, that any illumination is to be expected from a semantical theory. We can paraphrase Kreisel's 'principle of Christian charity' here: our only reason for saying that, when p and q are contingent-propositions, 'in every possible world in which p is true, q is true' is that we know that p entails q. There is no clairvoyance involved.

Nevertheless those logicians who prefer not to be troubled by the paradoxes are undoubtedly replacing our 'working' concept of entailment with some other concept, which seems to be more accessible in dealing with formalized languages. But it is one thing to know that some logically interesting interpretation makes, say $p \wedge \sim p. \rightarrow q$ trivially true, and another thing to accept that interpretation as some 'technical' sense of entailment. The fact is that the technical sense of entailment is well-esablished and it is reduced ad absurdum by the claim that $p \wedge \sim p. \rightarrow q$ is a 'true' entailment.

If the semantical approach does not lead to a decision concerning the status of the paradoxes of implication, what other alternative do we have? Formal logicians would undoubtedly say 'the syntactical approach', but then they must be careful not to smuggle in semantic concepts such as Tarski's 'satisfiability'.

Furthermore the issue has been confounded by C. I. Lewis' socalled independent proofs of the paradoxical formulae that kept popping up whenever the logician set out to formalize the notion of logical consequence. The proof, known already in the Middle Ages, is well-known and at first sight rather impressive. It purports to show that using modes of inference everyone will accept as valid, the paradoxes can be proven. The message is clear: either accept these modes of inference and the paradoxes that go with them, or you throw away the latter together with at least one of the former. Furthermore it is expected that the latter alternative cannot be consistently adopted without altering the notion of entailment one whishes to safeguard against the paradoxes.

To nobody's surprise some philosophers have been willing to pay the price and have found fault with every step in the Lewis-proof.

But it remains to be seen whether or not acceptance of the modes of inference to which Lewis appeals commits one to acceptance of the proof. This being said, it must be granted that the argument has the merit of not involving semantical concepts. The appeal is not to our concept of truth but directly to our concept of entailment. The proof for $p \wedge \sim p. \rightarrow q$ goes like this:

1) hypothesis: $p \wedge \sim p$;
2) since [a conjunction entails each of its conjuncts] $p \wedge \sim p$ entails both p and $\sim p$;
3) since [a disjunction is entailed by each of its disjuncts] p entails $p \vee q$;
4) since [entailment is a transitive relation] $p \wedge \sim p$ entails $p \vee q$;
5) since [a proposition entailing two or more propositions entails their conjunction] $p \wedge \sim p$ entails $\sim p \wedge (p \vee q)$;
6) since [the so-called disjunctive syllogism is a valid mode of inference] $\sim p \wedge (p \vee q)$ entails q;
7) since [entailment is a transitive relation] $p \wedge \sim p$ entails q.
   Q.E.D.

Square brackets enclose the appeals to our intuitions about entailment. I must admit that I accept all of them. But this does not commit me to accept out of hand the proof iself. There is indeed the question whether what the appeals justify is the step in the proof. To accept the argument is also to accept that this is he case. However I shall ry to argue that the consistency of the appeals excludes the possibility. That is, those who accept the proof and its consequences must as some point in the proof change their position about what they are doing.

Presenting some rules as appealing to our intuitions about entailment presupposes of course that we have intuitions about entailment. What, intuitively, is an entailment? Taking a naive view of logic – but not too naive – I would say that the proposition that q follows logically from the proposition that p, iff all that is meant when it is said that q is part of what is meant when it is said that p; or: the proposition that p includes the proposition that q; or: if S^p is the sentence used to assert that p and S^q is the sentence used to assert that q, the meaning of S^q is part of the meaning of S^p.

This talk of meanings should scare nobody. If a sentence has meaning, that meaning is a proposition, the logical form of that sentence. Thus, I fully embrace the old doctrine that entailment, logical consequence, holds in virtue of logical form..

It would be a welcome development if there were a satisfactory theory for the semantic interpretation of sentences, for then we could give theoretical descriptions of propositions – whereas now we sometimes feel obliged to say that the meaning of sentence one is given by the meaning of sentence two. But this scarecely illuminates the observer about the meaning of either.

But, one can say, this is of little importance in propositional logic where we are only interested in the ways in which the meaning of a complex sentence may come to depend upon the meaning of elementary sentences. Usually we are content with talk about truth-dependencies of complex propositions on elementary ones, but this presupposes that all problems concerning meaning have been dealt with. Notably the symbolism of propositional logic is a formal language created with the intention of formalizing ways of truth-dependency. It is a logically interesting exercise because it enables us to relate the meaning of the whole to the meaning of the parts, e.g. in dealing with a sentence that can be used to assert a conjuctive proposition. But there is a special tag to this procedure. In propositional logic there are some wellformed expressions the logical form of which turns out not to depend at all on the logical form of its material constituents. And here we can easily be misled by our symbolism; for it is a small step to understand $p \wedge q$ entails p in this way: that from the expression $p \wedge q$ the expression q follows: But this is a misunderstanding.

We are only allowed this: the proposition identified as the conjunction of the proposition that p and the proposition that q entails the proposition that q – independent of the grammatical form of the expression used to designate that proposition. This is especially obvious in the case of, say $p \wedge \sim p$ and $\sim (p > p)$. The theory entails that both expressions do not, cannot, derive their meaning from the meaning of the material constituents, p and q; furthermore it entails that both are equivalent. Hence, there can be no doubt that, if the expressions can be used to designate propositions, they must designate the same proposition. And this is quite acceptable. For if we give the logical connectives $\wedge, \sim, >$ their usual interpretation, we arrive at the same conclusion. Not because, of course, 'the one is true whenever the other is, and vice versa' is true, but because both 'mean' exactly the same thing.

To deny this, is to invite the question what is meant by the one that is not meant by the other.

Yet another question has to be answered: if p is propositional sign, expressing propositional content, what is the propositional content of the well-formed propositional sign $p \wedge \sim p$?

Even so, the above analysis seems to suggest a shorter proof for the controversal formula $p \wedge \sim p. \rightarrow q$. Indeed

$p \wedge \sim p$ entails $q \wedge \sim q$
$q \wedge \sim q$ entails q
$\therefore p \wedge \sim$ entails q

Indeed there seems to be no choice but to accept that $p \wedge \sim p$ and $q \wedge \sim q$ express the same proposition, and hence to accept that the first step in the proof is validated given our intuitive understanding of what it means to say that one proposition entails itself. But it is obvious that once this is granted we cannot consistently maintain that the second step in the proof is admissible, because the proposition to the left of 'entails' is quite irrelevant, to the proposition to the right of it, the similarity of their propositional signs not withstanding – note that this similarity is deified when it is said that q and $q \wedge \sim q$ are both truth-functions of q; but the point is that whereas the meaning of the expression q depends on the proposition that q, this cannot be said concerning the 'meaning' of $q \wedge \sim q$.

It should be noted that there is at least one theory of entailment, Anderson's and Belnap's, which would rule out step one in the foregoing proof, on the ground of their meta-theorem: if $A \rightarrow B$ is a provable entailment then A and B share a variable. The prima facie plausibility of their result is assured by their remark that 'A formal condition for "common meaning content" becomes almost obvious once we note that commonality of meaning in propositional logic is carried by commonality of propositional variables' (p. 48) [1]. But do all our wellformed expressions in propositional logic carry meaning? And if so, can we compare the meaning of the propositional sign p with that of $p \wedge \sim p$ in the same way as with the meaning of q?

Now to say that $p \wedge \sim p$ entails p must presuppose another concept of entailment as contrasted with that presupposed by $p \wedge \sim p$ entails $q \wedge \sim q$. If nothing else, the relevance-condition which Anderson and Belnap are so anxious about seems better to comply with the latter entailment than with the former, where the 'common meaning content' is obviously nil.

But of course there would be no problem if entailment were a relation between propositional signs. That would, however, be another attempt at trivializing the concern over the paradoxes of implication, making entailment an arbitrary operation in an uninterpreted formal system.

But if entailment is indeed what justifies an inference, it deserves better. After all, in virtue of what could we say that one sentence entails another, if not in virtue of the propositions we can designate by means of them? That is, in virtue of their logical form. It is obvious that in order to develop a logic of entailment we should not be deceived by the appearances of the symbolism we adopt.

Now it may be somewhat byzantine to go at all this length just to voice an opinion over one type of formula. However, there is more at stake. Let me therefore turn to the interpretation of disjunctive formulae. Indeed the Lewis-proof presupposes an important choice here. If someone were to say 'Russell is the author of the Theory of Types, or my name is Napoleon Solo' the hearer who is aware that the speaker's name is Jones would immediately grasp that what the latter is asserting, stressing, is that Russell is the author of the Theory of Types. Under the circumstances, he could infer from the assertion that someone invented a theory known as the theory of types. Why? because he realizes that the speaker's name is not Napoleon Solo. So there should be no problem with a propositional sign such as $p \vee (\sim q \wedge q)$. The proposition expressed can be no less than the one expressed by mean of p. In other words, the standard interpretation of 'or' commits us to the following: $p \vee (q \wedge \sim q)$ has meaning just in case p has meaning and then it has the same meaning. But this presupposes that $q \wedge \sim q$ is to be excluded on a priori grounds; in order to say that $p \vee (q \wedge \sim q)$ means something we have to say that it has exactly the same meaning as p. And this allows us to write that $p \vee (q \wedge \sim q)$ entails p. This is rather exceptional since we cannot in general say that $p \vee q$ entails p. Indeed in the latter case that what is meant when it is said that $p \vee q$ may well be q, but in the former case that what is meant when it is said that $p \vee (q \wedge \sim q)$ cannot be $q \wedge \sim q$. To deny this is to invite the question what it is that is meant when it is said that $q \wedge \sim q$.

The importance of this remark is illustrated by Anderson and Belnap's[3] 'proof' that the socalled disjunctive syllogism rests for its validity on the fallacious entailment $p \wedge \sim p \rightarrow q$ [2].
Consider: $\sim p \wedge (q \vee q)$ entails q. In order to assess the validity of the disjunctive syllogism we have to ask whether or not that what is meant when it is said that $\sim p \wedge (p \vee q)$ includes what is meant when it is said that q. Clearly the antecedent cannot be used to designate a contradiction without destroying the view that $q \vee (p \wedge \sim p)$ has exactly the same meaning as q – which need not be contradictory. For this is undoubtedly a validating rule:

$$P \wedge (Q \vee R) \text{ entails } (P \wedge Q) \vee (P \wedge R)$$

But then we have [$\sim p \wedge (p \vee q)$ entails $(\sim p \wedge p) \vee (\sim p \wedge q)$] and this, together with [$(\sim p \wedge p) \vee (\sim p \wedge q)$ entails $\sim p \wedge q$] and [$\sim p \wedge q$ entails q], yields [$\sim p \wedge (p \vee q)$ entails q]. The absurdity of Anderson's and Belnap's contention that one legitimate way of interpreting '$\sim p \wedge (p \vee q)$' is as if it were '$p \wedge \sim p$', is best illustrated with an immediate consequence of it. It would force us to admit that, according to standard interpretation, '$p \vee \sim p$' might well be interpreted as $p \wedge \sim p$. Apparently we are committing ourselves to the view that a reasonable interpretation of propositional logic excludes the interpretation of a formula as not taking its logical form from the logical form of its 'material' constituents whenever the formula is neither a theorem nor the negation of a theorem. This view is confirmed by the requirement that in a dialogue each new entry should, if at all possible, be interpreted as assertion of a proposition, i.e. as adding something definite to the commitments of the speaker. But is can hardly be said that, if a statement were interpreted independent of the meaning of its parts, we could determine what it is that is added to the commitments of the speaker. For a general formulation of the requirement, see e.g.[4]. Any one who accepts Lewis' argument admits the validity of the disjunctive syllogism and therefore also the view that $q \vee (p \wedge \sim p)$, or: $\sim p \wedge (q \vee p)$, both entail q – that is to say that neither of the propositions can be interpreted as giving as an alternative proposition included in them: $p \wedge \sim p$. And this interpretation presupposes that $p \wedge \sim p$ cannot really mean anything, that *we* cannot really mean anything with $p \wedge \sim p$. Otherwise acceptance of $\sim p \wedge (p \vee q)$ entails q must, as Anderson and Belnap have indicated, presuppose acceptance of $p \wedge \sim p$ entails q; thus rendering the proof circular. Now if entailment were a relation between propositional signs it could hardly be argued that, say, $q \vee (p \wedge \sim p)$ could not mean the same as $p \wedge \sim p$. But if entailment is a relation between propositions this is excluded, for there is no genuine proposition corresponding to $p \wedge \sim p$ whatever p may be, while there is one corresponding to $q \vee (p \wedge \sim p)$, i.e. q, whatever q may be.

To summarize this discussion: I have assumed that entailment is a relation between propositions, holding in virtue of the meaning of a propositional sign. This presupposes a theory for ascribing meanings to sentences. However, in propositional logic we are only concerned with the question of how the meaning of complex propositional signs can come to depend upon the meaning of its elementary (material) propositional signs. The problem was that in the language of propositional logic there are expressions that are independent of the meaning of their constituents.

In the language of truth-values: their truthvalue is independent of the truthvalue of their components. Inquiring into the propositions that are supposed to give the meaning of these expressions, there can be no criterion for discriminating among them (I have especially taken the case of contradiction, which is the clearest) and so, if we are to allow such expressions in the context of entailment, we have to say that they entail one another. Next we had the case of disjunctive formulae, especially where one of the disjuncts was logically false. Here too we had to assume that the propositions rather than the expressions were what mattered most. In both cases the conclusion was the same: $p \wedge \sim p$ entails p is not justified by the innocuous rule that a conjunction entails its conjuncts. For, undoubtedly that rule is about conjunctive propositions rather than about conjunctive expressions. But a conjunctive proposition is a proposition and this implies that we should, at least in principle, be able to determine what would be the case if it would be a true proposition.

The proof of P entails $q \vee \sim q$ can be treated along similar lines; it seems that if we were to try to shorten it to

$$q \rightarrow q \vee \sim q$$
$$q \vee \sim q \rightarrow p \vee \sim p$$
$$\therefore q \rightarrow p \vee \sim p$$

only the first step is eligible for critique. In this particular case we can quote Anderson and Belnap when they set out to criticize $p \rightarrow (p \rightarrow p)$ (p. 44) [1]; and this brings us to the final point: some of the expressions in our symbolism are said to be logically true, others said to be logically false. But there is no acceptable sense of entailment allowing us to infer logical truths from contingent propositions, or the latter from a logical falsity. For to deny this is to say that what we call logical truths and falsities somehow take their meaning from material components, and this is surely an absurdity.

Or as Anderson and Belnap would say: a modal fallacy.

In the Lewis-proof for q entails $p \vee \sim p$ we must therefore find fault with the steps which are incompatible with the notion of entailment as a relation between propositions:

q
$(q \& p) \vee (q \& \sim p)$ (1)
$(p \vee \sim p) \& q$ (2)
$(p \vee \sim p)$ (3)
$q \rightarrow (p \vee \sim p)$ (4)

It is clear that the premiss of the third step is the proposition that q – even if the propositional sign is a mere inflated version of the propositional sign q. It follows that acceptance of the third step in the proof leads to a circular proof, since it presupposes what we have to prove.
The upshot is undoubtedly that any appreciation of the entailment-paradodoxes must take account of the difference between proposition and propositional sign.

Literatuur

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