

MATHEMATICS, THE COMPUTER REVOLUTION  
AND THE REAL WORLD

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It is widely agreed that, the foundationalist program being exhausted, it is time for new directions in the philosophy of mathematics. Unfortunately, the plural in "directions" has been too rarely justified. We have seen a series of sociological case studies of the discovery of proofs in such areas as number theory, logic and the foundations of calculus. These have been not without interest, but are their choices of subject-matter perhaps too narrow? Sociology, like such sciences as psychology, history and anthropology, is a two-edged sword (or, perhaps more accurately, a boomerang). For these sciences are empirical (at least potentially), and subject to certain norms and standards of procedure and evidence. When Joseph de Maistre remarked, "You might as well say sheep are born carnivorous, and everywhere they eat grass", he exposed the fact that Rousseau was not an actual anthropologist, and that the Noble Savage was an inhabitant of the European mind's stock of paradigms, thought experiments, rational reconstructions, or myths, not of Tahiti.

If philosophers of mathematics wish to embrace the social sciences, they must take account of the appropriate methodology - and this means the current methodology, not that of speculative forerunners like Sir James George Frazer or Franz Anton Mesmer. If history is to be studied, it must be actual history, not reconstructions of it, rational or otherwise. If the psychology of mathematicians is to be used, a fair sample of mathematicians must be interviewed, and the investigator must be able to explain what experiments or surveys have been conducted, and at what level of significance were the results. A sociological or anthropological study must design and report its studies in the field in the standard way. In particular, with all such sciences, strict attention must be paid to whether any case studies used are "typical".

Now "being typical" is not an absolute term, but a relation

between an individual and a set of which it is a member. The reader may claim to be a typical late second millenium intellectual or a typical Sagittarius, but is certainly not a typical land-mammal. Some of the recent case studies undertaken in the philosophical literature might not be too atypical if the aim were to study the discovery of important theorems in pure mathematics by great mathematicians. Though even here, there would be doubts about representativeness - Dieudonné remarks that the central topic of twentieth century pure mathematics is algebraic topology,<sup>1</sup> which has a character quite different from areas more familiar to philosophers, like number theory and logic. But is theorem-proving by great mathematicians the natural, or only, object of study? Kings and battles once occupied a large part of history; picturesque as they still are, they do not absorb the attention of historians as exclusively as heretofore. As in history, so in mathematics, the humble toilers in the vineyard have their story to tell. They may have less access to the written record, so that the historian may need to work harder to hear their voice, and may have to integrate the individual stories with an approach *en masse*, but some such study is essential if the picture of the terrain is to be balanced. The theorem-provers and concept-stretchers are not the typical dwellers in the world of mathematics. The industrial mathematical modeller and systems designer, the engineering student, the stockmarket analyst, the educator, the cabalist and the computer scientist have quite different views of mathematics, and it is not clear why their views should be devalued *a priori*, if the aim is to understand what mathematics really is.

It is especially unclear why philosophers concentrate on pure mathematics at the expense of applied. Is it a residual Platonism that sees mathematics as a collection of theorems, and its relation to the world a poorly-explicated "idealisation from" and "application to", left in the hands of a class of *untermenschen* called physicists, operations researchers, defense contractors, etc.? Perhaps, but philosophers are generally vocal enough about the evils of Platonism and the need for other people, at least, to take practice seriously. Or is it that philosophers are in the nature of the case usually pure thinkers, and in their study of mathematics naturally gravitate towards the pure, the harmonious, the unsullied? Certainly few philosophers (Wittgenstein excepted) have had experience in applying mathematics, and few (Körner being a notable exception<sup>2</sup>) have taken applied mathematics seriously in their philosophy of mathematics. It must be admitted too that in the first half of this century mathematics was in a particularly "pure" phase, and that this has been

reflected in mathematics courses, which have been almost entirely composed of pure mathematics, plus assorted "methods", up to middle tertiary level. Even philosophers who have studied tertiary level mathematics have therefore had little exposure to a coherent body of applied mathematics. Since 1950, however, the pendulum has been swinging in the applied direction.<sup>3</sup> Some of the details and consequences of this will be discussed below.

In the meantime it would be appropriate for the philosopher to adopt a neutral stand on the pure/applied distinction. Perhaps it is correct after all to regard mathematics as a body of essential pure knowledge, whose relation of "application" to the world is to be explained. The "unreasonable effectiveness of mathematics"<sup>4</sup> is then an outstanding problem. On the other hand, perhaps the correct view is the one suggested by history, that mathematics is in the first instance applied, and pure mathematics consists in the hard problems of applied mathematics, which, after resisting solution in a single lifetime, acquire a life of their own - this being understood in a purely sociological sense, in that they are worked on by people who have forgotten their motivation.

In choosing a case study in the development of modern mathematics, it is therefore desirable to have something reasonably neutral between pure and applied mathematics.

The impact of computing on mathematics makes a good case study for several other reasons. It is the most obvious major change in mathematics in this century. The general public believes that computers are basically machines that do mathematics, and suspects that they have actually rendered mathematicians redundant. Controversy about the nature of the change extends to the mathematics profession too. It has been to the advantage of certain mathematicians to encourage the view that computing has "revolutionised" mathematics, since grants for computing are several orders of magnitude larger than those available for merely sitting and thinking. On the other hand, some pure mathematicians believe that computing is taken up by burnt-out colleagues, who thereafter spend their time solving trivial programming problems internal to their software, and never seem to return from the game world with anything of "real" mathematical interest. Philosophers have been ready to align themselves, at least *de facto*, with this reactionary clique. Apart from the Tymoczko-Levin exchange on the nature of computer-assisted proofs,<sup>5</sup> there has been remarkably little strictly philosophical interest in the impact of computing on mathematics. Surely there is a need for some examination of what effect computing has actually had on mathematics, and a con-

sideration of what this means for the philosophy of mathematics. In particular, if it is true, as certainly appears to be the case and is widely believed, that computers are doing things that were formerly part of mathematics, then philosophy will have to allow mathematics to include things that computers can do, and philosophical views which do not allow this will be in trouble. Presently-existing hardware cannot, for example, access the world of Platonic forms, nor can it await insights on the Continuum Hypothesis. On the other hand, if what computers are doing is not really mathematics, plainly there is a good deal of confusion and misconception to be sorted out.

Let us first evaluate the claim that computing has permeated and revolutionised all, or almost all, parts of mathematics. Of course it is true that the mathematics sections of libraries are full of journals of computer-related mathematics; computational mathematics and so on. But the fact that a group of academics sets up a *Quarterly Review of Cinematic Ontology*, and fills it with articles praising one other extravagantly, does not show their discipline is important, or that anyone else is taking notice. Perhaps they acquired tenure in the sixties, when the universities were expanding, and now they are attracted to an easy subject. To draw conclusions with any pretence to objectivity, one must survey something representative, such as the productions of mathematicians recognised by their peers as making major contributions, or the contents of the flagship journals of mathematics. Let us take, for example, the research-expository articles in the *Bulletin of the American Mathematical Society*. These are up-to-date surveys of what is happening in mathematics. They are written by the top people; they are wide-ranging; they are carefully selected for their interest to the mathematical community. From the time a new series of the *Bulletin* began in 1979 until April 1986, there were 106 of these articles. Of these, only six appear to have any relation to computing. One was on mathematical typography, another 'Computers and the nature of man: a historian's perspective on controversies about Artificial Intelligence', another 'Computer modelling of scientific and mathematical discovery processes'. These are interesting, and show an admirable breadth of vision in the editors, are not what one had in mind in talking about the transformation of mathematics by the "computer revolution". That leaves only three out of the hundred that could be called the real thing: one on the finite Fourier transform, which would be of little interest without computers, one on 'Recent developments in information-based complexity', and one by Stephen Smale, 'On the efficiency of algorithms of analysis'.<sup>6</sup> This last

represents the rare case of a noted mathematician changing direction under the influence of questions coming from computing. A similar story could be told of the winners of mathematics' most prestigious prize, the Fields Medal. Computing-related work is not totally absent, but the overwhelming majority of the winners are traditional pure mathematical theorem-provers. And among the revered ancestors from the ancient past of mathematics (i.e., the mid-century) only von Neumann and Turing had genuine applied and computing interests, and even they would undoubtedly not be remembered if they had not achieved first-rate results in pure mathematics as well.

I looked through some of the major journals of mathematics (that is, of "mathematics" simply, not computational mathematics or other specialities) for 1986. One is hard put to find more than a passing reference to computing, or a problem motivated by computing, in any of the 250 or so articles in *Inventiones Mathematicae* vol 86, *Journal of Algebra* 102, *Journal of the London Mathematical Society* 33, *Functional Analysis and its Applications* 20, *Illinois Journal of Mathematics* 30, or *Journal of the Australian Mathematical Society* A 40.

A few exceptions need to be noted. Number theorists have used computer packages to help their work,<sup>7</sup> and in some cases have used the computer's high speed of computation to collect large amounts of evidence for conjectures as yet unproved.<sup>8</sup> (It could be suggested that the reason for this is that number theory is very difficult, so that all problems in it that are both tractable and interesting have been solved, leaving the averagely-intelligent number theorist to find something else to do). Likewise, there have been certain discoveries of agreed merit which have used or been motivated by computing - the discovery of solitons, the proof of the four-colour theorem, the analysis of the average behaviour of algorithms, the theory of computational complexity, and the construction of certain of the sporadic finite simple groups (though even in the last case, the largest of these groups, the "Monster", was too large for a computer and had to be done by hand).<sup>9</sup>

The list is not long, and the recurrence of a few standard examples (what might be called the *Archaeopteryx* phenomenon) always engenders suspicion. The reason why the promise of computing has been so little fulfilled needs to be considered, but answers at this stage must remain purely speculative. It appears, for example, that the gap between the limit of hand calculations and the limit of computer calculations is not very large, in some conceptual sense. Less Platonic and more sociological reasons also suggest themselves. Perhaps mathematicians

are just not very adaptable people, and, like Galileo's opponents who are alleged to have refused to look through the telescope, have declined to avail themselves of the new technology. This explanation cannot be discounted too quickly, given that mathematics departments in the western world are almost entirely staffed by people of more than twice the age of peak mathematical productivity. But the main explanation may lie not so much in the lack of fulfillment of the promise as in the promise itself. Since the 1950's, the "computer revolution" has generated a stream of farcically overwrought promises (of "Artificial Intelligence within five years" and so on) whose continual refutation has only increased the effort (and of course money) devoted to their completion.<sup>10</sup>

Let us then draw the conclusion (being careful not to overstate the case):

Pure mathematical research has been only marginally affected by the "computer revolution".

Granted this, it does not follow that we should draw such a wider conclusion as, "Computing is nearly irrelevant to mathematics". For one thing, "irrelevant to" is an approximately symmetrical relation. If computing is irrelevant to pure mathematics, perhaps pure mathematics is irrelevant to computing, and hence irrelevant simply. Medieval logicians felt that scientific inference was irrelevant to them, but it was not scientific inference that ended on the scrapheap of history. Certainly this is exactly the conclusion that *students* have been drawing (and, as remarked above, their view is not to be dismissed without reason). In 1972 Rosser said, "Unless we revise [mathematics courses] so as to embody much use of computers, most of the clientele for these courses will instead be taking computer courses in 1984."<sup>11</sup> In the twelve years covered by the prediction, U.S. undergraduate and graduate degrees in mathematics fell by 50%. Lynn Steen writes:

Departments that have a strong traditional major often fail to provide their students with the robust background needed to survive the evolutionary turmoil in the mathematical sciences. Like the Giant Panda, these departments depend for survival on a dwindling supply of bamboo - strong students interested in pure mathematics.<sup>12</sup>

(The philosopher will ask, "What are these 'mathematical sciences'?)")

The mathematics profession has therefore an input from the outside at one point - in its teaching (from which derives most of its money). The view of the mathematical world is one thing for the gods on Olympus (the referees of *Inventiones*, say); it is quite another for the embattled head of department fighting exponential decay of graduate numbers. Down in the bunkers, a defence of mathematics is being mounted which involves a quite different view of the subject from that familiar to philosophers. Krantz, for example, advises students that they are being duped into taking computing majors, when computing really consists of some trivialities plus mathematics:

The ability to write a program requires that one be familiar with certain patterns of logical thought, and that he memorizes a few details of a particular programming language. The latter are trivial, while the former are the stock-in-trade of the mathematician. Math majors are not only trained in a number of advanced mathematical techniques, but they are also trained in abstract reasoning, modelling and problem solving.<sup>13</sup>

While philosophers often regard mathematics as almost all about proofs, their discovery, refutation and completion, the truth on the ground, in "actually taught mathematics" (and even more in actually learnt mathematics) is almost the opposite. The majority of tertiary mathematics students cannot prove anything whatsoever, and tertiary exams of necessity adapt to this fact by not requiring them to do so, and allowing a pass for competence in applying set rules, formulas and algorithms to standard problems. In fact, this is the sort of thing that a computer can do much better, and one can indeed buy a good hand-held calculator which is capable of passing the average first-year university calculus exam on its own. Mathematics teachers who suggest that students need to know techniques of proof are found having to justify their case very carefully,<sup>14</sup> and relying on support from Computer Science professionals, who have long since realised that computer programs are logical objects, and require first and foremost skills in logic for their study.<sup>15</sup>

The historical fact of the 1980's, then, is that mathematically able students take Computer Science, while less able students enrol in mathematics and force academics to teach them algorithms. This is real mathematics, not philosophers' rational reconstructions. What does this mean for the future of mathematics? Let us draw an analogy.

When wheels were invented, there grew up a highly respected

profession of Wheel Scientists, whose job was to follow the wheels, make sure they stayed round, put them back on when they fell off, and advise on the tasks for which wheels could be used. Everyone wanted their children to become wheel scientists and axle systems managers. The students who got the highest marks at school all enrolled in Wheel Science. The rest enrolled in Cyclometry, where those who could grasp the concept of a radius were given A's and allowed to transfer to Wheel Science. But eventually, some of the wheel scientists were so good, that they found a way of making cheap wheels that stayed round, and never fell off. Everybody used these wheels, even if they had not studied Wheel Science. There was much public concern at first that young cart operators and charioteers were driving around with almost no idea of Wheel Science at all. But soon it was realised that the skills of driving and those of Wheel Science were totally and absolutely disjoint. Before long, there were no wheel scientists left. There were only a few poorly paid wheel engineers, who fixed chariots for people who couldn't afford new ones, and a number of highly paid Wheels Systems Consultants, who sold people more expensive systems than they needed.

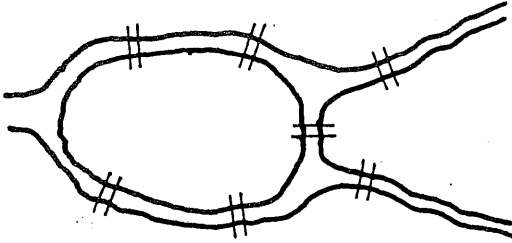
It is the same with computers. We have now reached the stage where very powerful and very accurate packages are interfaced with user-friendly expert systems, if necessary menu-driven. They give the right answer all the time, or warn if they cannot. Anyone who can ask the right question can get the right answer. If what the package is doing at the moment is not what is wanted, one does not fiddle with the code, one chooses another option, because packages are very flexible. It is unnecessary to worry about the mathematical underpinnings, because the package itself attends to such problems as error propagation and convergence. Students who use calculators as an infallible black box are wrong today, but they will be right very soon. To use packages well, what one needs is skill in deciding on the right question, that is, in analysing the problem. The mathematical skills involved are from such areas as modelling and simulation, operations research and statistics. One may need also, say, legal knowledge and communications and political skills, so that the answers are actually usable. The one thing not needed is any knowledge of computer science.

The philosophy of mathematics must, then, take account of the fact that the centre of gravity of mathematics is not now (if it ever was) in theorem-proving or number theory, but in industrial and business applications, using substantial computing (but using skills that are fundamentally mathematics, not computer



science in the strict sense). From this point of view, the question of what effect the computer revolution has had is again wide open. Plainly computing is allowing mathematicians to do kinds of things impossible before, and to do old things in new ways. What things? That is for the philosopher to say, and meditate on.

Close concentration on a case study will show how computing can transform the philosophical significance of a mathematical problem. Consider the old problem of Euler on the bridges of Königsberg:



The citizens of Königsberg noticed that it seemed to be impossible to walk over all the bridges, without walking over at least one of them twice. Euler proved their conjecture correct with an ingenious proof<sup>16</sup> (which has never needed any assistance from concept-stretching, monster-barring or other approved aids; it cannot have counterexamples either, since only universal generalisations have counterexamples, and Euler's proof applied in the first instance to a *particular* arrangement of bridges). His proof enjoys many mathematical perfections, especially that of generalising easily to answer the same question for *any* arrangement of land and bridges. Nevertheless, there is another and more obvious method of attacking the original problem: generate (by hand or computer) all the paths which do not go over any bridge twice, and check whether any of them goes over all the bridges once. The number of such paths is not large (certainly less than a few thousand), and the checking can be done straightforwardly and quickly by computer. Or slowly, but still straightforwardly, by hand. Now any mathematician in his right mind will of course prefer the simple and elegant proof of Euler to the long slog of brute checking. But the philosopher will take note of the fact that the brute checking method has answered

the *same* mathematical question as Euler. The (mathematical) question is the same, the (mathematical) answer is the same, the degree of certainty of the answer is the same. The aesthetics is different. The minimal conclusion the philosopher should draw is that logic and proof, in the traditional sense, are inessential to mathematics, and are replaceable by computation. The task then is to explain what is *mathematical* about the question and the answer.

Furthermore, the use made of computational power in this example is typical. Euler's problem is unusual in admitting of a neat solution by proof. The typical problem involving paths through networks, the allocation of resources, timetabling, or efficient design is solved by first using reasoning to eliminate some of the possibilities, and then searching the remaining set of possibilities by computer. (The four-colour theorem was also proved in this way). Similarly for the use of computing in areas formerly approximated by classical mechanics. The "finite element method" of stress analysis<sup>17</sup> proceeds by regarding a ship (say) as cut into many small rigid cubes joined at the corners, and computes the stress accordingly; similar methods apply for the computation of planetary orbits,<sup>18</sup> the interaction of ocean waves, and so on. The classical "analytic" methods were conceptually attractive, but were restricted to the simplest shapes of ships, orbits, etc.; arbitrary complications simply made the proof inapplicable. Computational methods can deal with in principle arbitrary complications, like funnels on ships and moons around planets; one just adds them to the calculations and lets the computer proceed. Much the same has happened in statistics, where non-parametric methods have tended to use brute computation to replace older methods involving simplifying assumptions. The relevant "finite" or "discrete" mathematics, including such topics as graph theory and combinatorics, has taken its place in curricula, and the large American publishers are producing the usual stream of textbooks on the subject, identical except for their increasing weight.

The philosopher should not perform his usual trick of dismissing these developments as only further masses of details, with nothing new "in principle". It is not true that if one finally understands  $7 + 5 = 12$  and the Axiom of Choice, one will understand finite mathematics "in principle". The point is that computational methods are simple and direct. The old apparatus of approximating the discrete by the continuous and then using the mathematics of the infinite, or infinitesimal, is done away with,<sup>19</sup> along with the equivalent of questions like, "What is the relation between the perfect mathematical sphere and the imper-

fect bronze sphere?" Computer simulation and mathematical modelling can mimic at least finite systems directly, and thence draw conclusions that apply *directly* to the real world. Once the computer has checked all the paths in the bridges of Königsberg problem, a mathematical fact is known, not about some idealisation, or logical possibility, or Platonic form, but about the bridges of Königsberg. Computers have brought back the conceptually simplest, most directly applicable and hence most philosophically interesting kind of mathematics. Let the philosopher take such mathematics as his paradigm.

In the first half of the century, the philosophy of mathematics tended to become fixated on a single question: What is the foundation of mathematics? Since 1960, new work has tended to fixate on a different question: Does mathematical proof lead to certain knowledge? These questions are reasonable, but have surely been debated enough. In the hope of encouraging some pluralism in the subject, I propose 25 research projects in the philosophy of mathematics:

- (1) Define precisely "structure" and "pattern". Does mathematics study them/it?
- (2) What is the logical status of statistical inference?
- (3) Is geometry mathematics or physics?
- (4) What do the results of psychologists on the cognitive development of mathematical abilities show about the nature of mathematics?
- (5) What is the role and logical status of experimental evidence in mathematics?
- (6) Are operations research and systems engineering mathematics?
- (7) Attach some meaning to the claim, "Mathematics is the language of science".
- (8) Account for the "unreasonable effectiveness of mathematics".
- (9) What is quantity? Does mathematics study it?
- (10) Has mathematics been unreasonably ineffective in social sciences? In meteorology?
- (11) What is complexity? Can it be measured? Is it the business of mathematics to answer a question like, "Could something as complex as *homo sapiens* have evolved by chance mutations and natural selection in 4 billion years?"
- (12) Sort the wheat from the chaff in catastrophe theory, chaos theory, fuzzy logic and fractal geometry.
- (13) How much of mathematics can be defined using only the notions of part and whole (and logical concepts)?
- (14) Why do sciences become more mathematical as they develop?
- (15) What aspects of reality exactly can be captured in a mathe-

mathematical model or computer simulation? What does the model or simulation have in common with the modelled?

(16) What is the logical and epistemological status of mathematical insight?

(17) What general restrictions does combinatorial explosion place on the computational power of intelligences, both natural and artificial?

(18) What are the scope and limits of computer methods and algorithms in mathematics, Strategic Defense Initiatives, etc?

(19) What mathematical problems are interesting? What makes them so?

(20) What are symmetry, continuity and order? Does mathematics study them?

(21) Are there mathematically necessary statements that also apply to the real world?

(22) What exactly is the difference between mathematics and science which allows mathematics but not science to be done in the armchair?

(23) What is the relation between mathematics and logic?

(24) Define "random", so as to make sense of the question, "Are the first million digits of  $\pi$  random?"

(25) What are the morally permissible, and what the morally worthwhile, directions in which a person with mathematical abilities should deploy them?

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## NOTES

1. Dieudonné (1982), p. 7.
2. Körner (1962), especially ch. 8.
3. Gray (1985).
4. The famous phrase from Wigner (1960).
5. Tymoczko (1979), reprinted in Tymoczko (1986); Levin (1981), not reprinted in Tymoczko (1986).
6. Smale (1985).
7. For example, the articles in Atkin and Birch (1971), and Andrews (1984).
8. Examples and philosophical discussion in Franklin (1987).
9. But see further Steen (1988).
10. See generally Dreyfus (1972), Weizenbaum (1976), Roszak (1986).
11. Rosser (1972), p. 639.
12. Steen (1986), p. 54.

13. Krantz (1984).
14. Franklin and Daoud (1988), preface and ch. 1.
15. Shaw (1984), p. 81.
16. Euler (1953).
17. See Williamson (1980), with further references.
18. A philosophical analysis in Franklin (1989).
19. Ralston (1981).

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