

MATHEMATICAL SKEPTICISM: ARE WE BRAINS IN A COUNTABLE VAT?

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In the anthology *New Directions*, I espoused the cause of quasi-empiricism as an approach in the philosophy of mathematics, borrowing the term from Lakatos and Putnam.¹ There are two key aspects to quasi-empiricism: first and foremost, an emphasis on mathematical practice, and second, an openness to scientific methods in mathematics such as computer proofs. Although these two features are compatible, nevertheless, they can pull in opposite directions.

The second feature, which invites us to see mathematical methods as close to scientific methods - or closer than was once thought - goes naturally with a realist view of mathematical objects and the image of mathematicians as discovering the nature of mathematical reality just as scientists are imagined to discover the nature of physical reality. This view is most natural with regard to computer proofs like that of the Four Color Theorem and the recent proof that there is no finite projective plane of order 10.² In such cases, it is easy to believe that there is a mathematical object, a 'formal proof' (actually a mathematical pattern of a certain sort), whose existence we become aware of by empirical means. As one of the 'authors' of the latter proof, Clement Lam, said "in this kind of problem, the mathematician can not personally check each step of a complete proof. But with computers, mathematical proofs are becoming more like the experiments conducted in the physical sciences than the traditional proofs of Euclidean geometry. When you get a result with a computer, the best you may be able to do is to show that the same result will be obtained if someone else repeats the experiment in just the same way."³

However, this scientific strand of quasi-empiricism is strained when it is extended to other branches of mathematics, especially to transfinite set theory. In a recent article, Penelope Maddy attempts to make the case for quasi-empirical methods in evaluating potential set theoretic axioms such as the Continuum

Hypothesis or Measurable Cardinals.⁴ In my opinion, the attempt is technically flawed since the quasi-empirical methods she discusses are all seriously inconclusive: probabilistic arguments for a given axiom are balanced by probabilistic arguments against it. Thus, there is really no analogy to science where probabilistic arguments actually do carry the day. More significantly, this quasi-scientific picture of set theory makes sense only if we adopt a strong realist stance to the existence of a determinate universe of sets. It differs from classical realism, or platonism, only by allowing that our intuitions of this mathematical reality might not be exhausted by traditional proofs, but can be supplemented by probabilistic arguments and inductive evidence. It is at this point that the tension with the emphasis on mathematical practice becomes most apparent.

In the *New Directions* anthology, I identified quasi-empiricism in the philosophy of mathematics as that approach which begins with and seeks to understand the actual practice of mathematicians. One motive for this was to free philosophy from a priori conceptions of what mathematics must be like – especially from the conception that mathematical results must be a priori, but also from other foundational programs. This emphasis on mathematical practice is naturally seen as an empirical or sociological approach to mathematics, an approach which Rene Thom characterizes, with some justice, as the view that “a proof P is accepted as rigorous if it obtains the endorsement of the leading specialists of the time.”⁵ This view of mathematics as a human practice or a cultural institution is championed by mathematicians such as Wilder, Davis and Hersh, philosophers, especially Wittgenstein, also Lakatos, Bloor, myself, and others such as De Millo, Lipton and Perlis. The more one focuses on the particularities of the mathematical community and its practices, the less one focuses on an alleged reality that they are uncovering. Wittgenstein was especially adamant in his attacks on realism or platonism in mathematics and in his insistence that mathematics was essentially different from science.⁶

Thus the first moral I draw is that quasi-empiricism, like politics, makes strange bedfellows. Many people can claim, or be claimed under, the banner of quasi-empiricism, but the differences among us can be quite dramatic. In the essay that follows, I will make a tentative stab at sorting out some of these differences. I will focus attention on one branch of mathematical practice, contemporary set theory, but I will treat it as a branch of mathematics like any other, ignoring claims that set theory is a foundation of mathematics. Does this practice commit us, or its practitioners, to a belief in an independent universe of sets and

if so, what further commitments does that belief entail? My strategy will be to recreate in set theory a version of the classical skeptical dilemma in epistemology. In the end, I will hazard some conclusions about the 'sociological view' of mathematics, but the value of the essay, such as it is, is in the connections it makes along the way to those tentative conclusions.

Skepticism is a central topic of modern philosophy. Yet although skepticism poses a formidable challenge to empirical knowledge and to knowledge of other minds, one does not often read of mathematical skepticism as a challenge to mathematical knowledge. There are several reasons for this lack.

In the first place, skepticism turns on a distinction between appearance and reality. The evidence for our beliefs is derived from how things appear to us and the truth of our beliefs is determined by how things actually are. Skepticism begins by nothing the gap between the two, but in the case of mathematics, it seems that somehow appearance and reality are one and the same. A dreamt proof is still a proof.

"Whether I am awake or asleep, two and three add up to five, and a square has only four sides; and it seems impossible for such truths to fall under a suspicion of being false."

In the second place, by attacking mathematical knowledge, the skeptic is in danger of biting off more than he can chew. Mathematics is so close to rationality, to the very possibility of logical discourse, that if it is called into question, the very framework of questions and answers is undermined. It is true that Descartes went on to ask "may not God likewise make me go wrong whenever I add two and three, or count the sides of a square, or do any simpler thing that might be imagined?", but he wisely does not pursue the point. No matter how convincing a reply we could construct - let it have the full rigor of a mathematical proof - the skeptic could counter that for all we know it only seems convincing to us and in reality we are 'going wrong'. Philosophical skepticism, as opposed to blind irrationalism, presupposes we are rational and that is pretty close to presupposing mathematics. Without this, the skeptic's game is over before it has even begun.

A third reason for the absence of mathematical skepticism might be that it is not really absent at all but goes by a different name. On one reading of traditional skepticism, it is not so much empirical knowledge that comes into question but the

interpretation of that knowledge: our claims to knowledge of a physical world existing independently of our perceptions. An analogous version of mathematical skepticism would not question mathematical knowledge, but our claims to know of a mathematical reality existing independently of our proofs. Thus construed, mathematical skepticism turns out to be anti-platonism or anti-realism, the denial of a mathematical reality independent of us, and so both formalism and intuitionism might be seen as versions of mathematical skepticism.

Nevertheless I am unhappy with the way this analogy leaves things. If one is, as I am, dissatisfied with both formalism and intuitionism for the many cogent reasons given in the literature, one seems to be driven back to platonism and the issue of skepticism is swept under the rug. More precisely, platonists attempt to avoid skepticism by postulating a faculty of mathematical intuition which is to insure our direct contact with the mathematical realm. However when the matter is put in terms of skepticism, the appeal to intuition seems especially question-begging. No one who engaged with the traditional skeptic about our knowledge of the external world or of other minds would accept as a solution that we just intuit the external world or other minds - as if we needed only to introduce a word to defeat skepticism!

Let me return a moment to the first two reasons given above for the lack of discussion of mathematical skepticism. These were our apparent inability to distinguish between appearance and reality in mathematics and the essential connection between mathematics and rationality. I want to suggest that these reasons do not apply with equal force to all of mathematics. I concede their validity for some branches of mathematics such as number theory. But I will contest their validity for contemporary transfinite set theory. However plausible it is that first order logic and number theory are constitutive of the notion of rationality, it is not at all plausible that the Axiom of Choice, Continuum Hypothesis, or even Replacement are essential to rationality. Moreover, by developing a version of Skolem's Paradox, I'll attempt to draw the contrast between appearance and reality in mathematics - specifically by suggesting that 'for all we know' we might really be living in a countable world. This way of putting 'the paradox', besides being colorful, severely tests some purported resolutions including appeals to mathematical practice. After examining some of these, I'll present a solution based on Putnam's resolution of the traditional brains in a vat dilemma and conclude with some general comments about quasi-empiricism.

Traditional skepticism, at least one form of it, begins with the distinction between appearance and reality. More exactly, it makes the claim that what justifies our ordinary beliefs must be 'internal' processes, experiences and ratiocinations, while what makes these beliefs true is their correspondence to features of external reality. The skeptic then goes on to argue that our inner experiences and thoughts could remain exactly the same, even though the outer world were radically different - eg., we could be dreaming or deceived by an evil demon. In a modern version, the skeptical possibility is expressed in the parable of brains in a vat. It is consistent with physical principles that the universe could have evolved a giant computer connected to brains kept in a vat of nutrient fluid in such a way that the computer feeds the brains just the elaborate systematic experiences we actually have! If this were so, it would seem that the totality of available evidence cannot tell between this alternative and the usual more comfortable picture of an enduring world of physical objects.

This form of skepticism has a serious point: it is that our overall picture of the world with ourselves in it does not hang together well. There is, in principle, a gap between our evidence and what we take ourselves to claim. So the skeptic's challenge to us is not 'to solve a puzzle' but to deal with the gap in our conceptual system he has apparently uncovered.

It is worth noting that skepticism does not depend on demonology. The gap can be brought out by taking our scientific picture of the world very seriously. Quine does this by starting with our best scientific picture of the world, then drawing a line around each of us at the limits of our sensory receptors. Any information we can get about the outside world, he claims, must be carried by energy transfers across this line. But then Quine notes that this evidence is very meager and does not uniquely determine a theory of the external world; in fact, the totality of all possible evidence radically underdetermines scientific theory.⁸ We cannot, as it were, get back to where we started by reasoning from the evidence. Now the brains in vat scenario is just one way to make this predicament graphic.

Given this way of looking at things, there are two natural ways around skepticism (three if we count Descartes' heroic attempt - he calls upon an omnipotent and all-loving God Who insures that our proper justifications ('clear and distinct ideas') correspond with reality). One path is to stick to justification and inner experience and to abandon external reality. So phenomenologists try to reconstruct our apparent talk of the external world as talk about sense data. The other path is to stick to

reality and truth and to abandon justification. So materialists, (in a current manifestation, reliabilists), argue that we're just bodies in a physical world and if our senses are reliable indicators of our environment, then that is all we can ask.

If mathematical skepticism is a cogent possibility - and that is yet to be established - we can draw some parallels with the above. Formalism (conventionalism, constructivism) is the obvious analogue of phenomenalism; each giving up truth and reconstructing reality in terms of justification. But platonism is a very poor analogue to materialism, it is so hard to draw causal connections between the unchanging world of mathematical reality and our occasioned beliefs. Rather, platonism is the analogue of heroic Cartesianism with mathematical intuition playing the role Descartes assigned to God. With this much as a background, let us attempt to reconstruct the skeptic's argument inside mathematics.

In his famous Diagonal Argument, Cantor proved that the real numbers are uncountable (in the process, he gave mathematical significance to the distinction between countable and uncountable sets).⁹ Later, Skolem showed that any consistent theory, including the theory which Cantor used for his proof, could be interpreted over a countable model. Thus a mathematician living in such a countable model could use Cantor's very argument to prove that the reals in her world were uncountable. From outside the model, however, we could see that the reals in it, indeed the entire model, form a countable set. This appears to be a paradox: it appears that the reals are both countable and uncountable. This appearance is sometimes called Skolem's Paradox.

But if it is a paradox, it is not a very serious one and can be dissolved by observing that a referential shift occurs when we pass from one model of a language to another (where 'language' is interpreted as a purely syntactical object). Because the countable world is a model for set theory, mathematicians in it can prove the same theorems of set theory that we can. Hence they can prove the sentence 'The reals are uncountable'. But in their usage 'the reals' refers not to what we refer to by the same word, but to a set in their universe. We know, and can prove by our construction, that that set is countable. However, by 'uncountable', they refer to the concept of uncountability in their model. They give the same *definition* of uncountable as we do - there does not exist a 1-1 function from the natural numbers onto the set in question. But this definition picks out a different property in their world, just as the world 'reals' picks out a different set. Their reals are uncountable in their world

because there is no appropriate 1-1 function in their world. Thus the apparent contradiction is dissolved. The reals really are uncountable. The reals in their model really are not the reals but a countable subset. However they have no notion of a reality beyond their model (that's what means to have a model or interpretation of their language). So what they mean by the sentence 'The reals are countable' is true in their world.

Having resolved one paradox, however, we find another problem arising to take its place: how do we know that we are not living in a countable model of set theory? The mere fact that we can prove that the reals are uncountable in our terms no longer seems sufficient since mathematicians in a countable model could apparently give the same proof in their terms! Does it make sense to suppose we might be living in a countable universe? Prima facie, yes, as much sense as it makes to assume we are brains in a vat. After all, couldn't God just erase all but countably many mathematical objects from existence without our noticing the difference? Or, if this puts too much strain on God's power, suppose the computer tinkered with the brains in a vat just a bit so that their faculty of mathematical intuition was altered. Whenever they attempted to grasp the reals (or the universe of sets) they succeeded only in grasping a countable subset (or a countable submodel). They would be brains in a countable vat.

Suppose, for a moment, that there is a real possibility that we might be brains in a countable vat. Would that matter? Some people are inclined to say no, just as they would to the standard skeptical possibilities. After all, if we can't tell the difference, why should it matter? But this is to ignore the conceptual challenge posed by skepticism. If we could be living in a countable world, then that possibility seems to undermine the distinction between countable and uncountable sets. As Skolem came to believe, it could be interpreted as showing that the distinction is relative, not absolute.¹⁰ Uncountable sets might not be *big* sets, just *complicated* ones. What would be threatened is our way of picturing mathematics, what Wittgenstein called mathematicians' 'prose'. It is not that we couldn't go on to do set theory in the same way given the skeptical possibility, but that once we recognized it, we might well not go on as we had previously. We might come to regard talk of large cardinals et al as excessively metaphorical, as puffed up. Mathematicians might come to believe, as Wittgenstein hoped they would, that Cantor's Paradise was really no paradise at all and so leave of their own accord.¹¹

Before examining some solutions to the skeptical puzzle let

me make a few brief comments on this mathematical skepticism. In the first place it nicely parallels empirical skepticism. As we saw earlier, it is possible to argue that a thorough going scientific picture of the world leads to a skeptical position - the evidence that we have, according to science, is insufficient to justify the science we began with. So too Skolem's skepticism can be seen as starting from a thorough going set theoretic picture of the world and leading to a skeptical position - the proofs that we have, according to mathematics, are insufficient to justify the mathematics we began with.

In the second place, notice that it is precisely a realist conception of mathematical entities that gets us into trouble. This is par for the course: as we saw in the beginning of this paper, skepticism trades on the distinction between what makes propositions true (relation to reality) and what justifies them (appearances, in this case proofs). Ordinarily one explains mathematics by assuming there is a real world of sets and that somehow or other (God's Grace) we latch onto the sets in the right sort of way. But then we notice that if we latched onto only a small subset of the totality; the countable sets, in the 'right sort of way' then everything would go on as before. We cannot tell the difference between the naive platonic connection and the perverse platonic connection. Thus the idea that we are 'brains in a countable vat' seems quite cogent, a very apt metaphor for the above predicament. And we have kept the promise of distinguishing between appearance and reality in mathematics. The appearance of an uncountable reality is quite compatible with an actually countable reality.

Lastly, I suggest that although realism gives rise to skepticism, there is no way around skepticism from a realist position. Appeals to mathematical intuition just won't help for the simple reason that mathematical intuition is equally operative for the brains in the vat. What distinguishes us from the brains is not the existence of a mathematical reality, nor a mysterious intuition connecting us to that reality, nor our ability to do proofs. All these we share with the brains in a countable vat. For the realist to defeat skepticism, he needs some further hypothesis, like a Cartesian God, who insures our intuitions get connected up in the right way.

Before presenting my own solution to Skolem's Paradox, let me briefly review Skolem's two solutions as discussed by Paul Benacerraf.¹² Originally Skolem presented his results as showing the inadequacy of axiomatic set theory as a foundation of mathematics. However, although the foundations controversy was raging in the early twentieth century, it is no longer very

relevant today. We can excise foundationalist concerns from Skolem's first answer rather easily to get the conclusion that formalization, axiomatization in first order logic, is insufficient to capture informal mathematics (in particular, the theory of transfinite sets). Benacerraf seems to endorse this answer. The Skolem-Benacerraf view is that pure formalization admits unintended models, so that the intended model can not be captured by this method.

This answer is quite familiar as one of the standard anti-formalist critiques based on various limitation theorems. Notice, however, that it is effective only on the assumption that Skolem's results do show something *prima facie* untoward about mathematics. If those results were unexceptional - in our terms, if there were no possibility that we lived in a countable world - then nothing is shown about formalism or anything else. Notice further, that this answer turns on a particular realist framework for formalism, that axiom systems are meant to characterize models and the failure of categoricity is the untoward result (this feature is not an objection to my development of skepticism, since that development, begins by assuming the concepts and distinctions it will later call into question). Finally, the Skolem-Benacerraf answer will only be completed if we can explain what it is that the formalist axiomatization leaves out. As I argued above, that explanation cannot be provided by general talk about a universe of sets and mathematician's intuitions about that universe for that general talk can be accommodated by the skeptic's example of brains in a countable vat. Moreover, it is hard to see how Benacerraf's appeal to 'informal practice' can solve anything. Exactly what is it about our informal practice that guarantees that the reals are uncountable in a way in which a formal version of Cantor's proof apparently does not?

Skolem's second answer, according to Benacerraf, is the one he arrived at later in life and the one usually associated with his name. Skolem Two has accepted the formal method as providing all that can be said about mathematics. Consequently, he concludes that such concepts as countable, uncountable are essentially relative. Exactly what Skolem meant by relative is open to question, but the most natural explanation is the one alluded to above. Metaphorically, we might say that the concepts are relative in the sense that the brains in a countable vat makes dramatic. To persist in a realist viewpoint is to commit oneself to mathematical skepticism; it is to make, eg., the reals or the power set of integers, relative to our starting point. On the other hand, if we take an extreme formalist position where the axiom systems are sufficient unto themselves (they are not attempts to charac-

terize models of any sort), then the term 'relativism' drops away as irrelevant, but all we are left with is a systematic and empty formal theory. The content of the countable-uncountable distinction reduces to nothing more than is given in the formal axioms. In particular, the view that uncountable sets are very large collections is nothing but a picture; there are no things outside the formal system itself to be large or small. (Here again we have an echo of Wittgenstein's attack on transfinite set theory: the extension from finite sets to infinite sets is totally unwarranted and only a colorful, but misguided, way of speaking.)

In his reply to Benacerraf, Crispin Wright offers the following possibility for preserving the countable-uncountable distinction from the skeptic's threat.

"... I believe we can glimpse the possibility of an alternative to the platonism which, pending disclosure of some internal flaw in the skeptical arguments, was all that seemed available as an alternative to the skeptical conclusion ... What we need to win through to, I suggest, is a perspective from which we may both repudiate any suggestion of the platonic transcendence of meaning over use and recognise that meaning cannot be determined to within uniqueness if the sole determinants are rational methodology and an as-large-as-you-like pool of data about use. Wittgenstein wanted to suggest that the missing parameter, the source of determinacy, is human nature. Coming to understand an expression is not and cannot be a matter of arriving at a uniquely rational solution of the problem of interpreting witnessed use of it - a 'best explanation' of the data. Still less is it a matter of getting into some form of direct intellectual contact with a platonic concept, or whatever. It is a matter of acquiring the capacity to participate in a practice, or set of practices, in which the use of that expression is a component. And the capacity to acquire this capacity is something with which we are endowed not just by our rational faculties but to which elements in our sub-rational natures also contribute: certain natural propensities we have to uphold particular patterns of judgement and response."¹³

But it is not so clear how to interpret the Wittgensteinian picture for the case at hand. It would make sense, to me at least, applied to the Gödel case of unprovable formulas. One likes to think of the concept of number as determinate; each closed formula or its negation is true. Now Gödel seems to have shown

that if this is so, no formalization can capture the intuitive conception of number. And the formalist's natural response, that there is intrinsic undecidability here, seems forced. On the other hand, the platonist's appeal to mathematical reality grounding our intuitions brings its own difficulties. But in this case, Wittgenstein seems to offer a genuine alternative in just the manner that Wright suggests. Both formalism and realism are wrong. But formalism is not wrong because it has failed to articulate beliefs (intuitions) about the numbers we possess implicitly. Rather it is because we are the kind of creatures that we are, because of human nature, that we respond in the same way to Gödel's result, that we all accept the unprovable formula as true and its unprovable negation as false. We come to count something as a proof despite the fact it cannot be represented in our formalism, but in so doing, we are not being faithful to some determinate mathematical reality independent of us, nor are we relying on some intuition the formalist forgot to formalize. We are going on in a genuinely new way and it happens to be the same way for all of us because we agree in 'our sub-rational natures'. Those who respond differently risk expulsion from the community of mathematicians.

Right or wrong, this picture has a certain plausibility when applied to Gödel, but how is it supposed to apply in Skolem's case? There doesn't seem room for a choice here I don't see where our practice, informal mathematics or subrational natures can take hold, can guarantee the countable-uncountable distinction is the one we really want and not some countable counterfeits. To be sure, I think there is something very significant in Benacerraf's and Wright's suggestions. These elements do contribute to our way of doing set theory and I will return to this later. But they cannot solve the dilemma of brains in a countable vat for the simple reason that all these elements can occur with the brains in a vat! (Unless, perhaps, one needs a body to do set theory, but I set this possibility aside.)

There is an answer to the Skolem problem and it is implicit in the work of Hilary Putnam.¹⁴ The marvelous thing about Putnam's argument is that it works equally well to show that we are not brains in a countable vat and to show that we are not brains in a vat of any sort! He is willing to grant, for the sake of argument, that there could be brains in a vat (countable or not). The thrust of the argument is that we are not they; just as I might grant that there are possible worlds in which I don't exist, but still argue that ours is not such a world. One of Putnam's key insights is that the brains in such situations are not deceived about anything. They aren't deceived because their

words don't mean the same as ours (if they did, at least one of us would have to be systematically deceived). But as we saw in describing the Skolem case, there is a referential shift that occurs when the same syntactic language is reinterpreted over the vat (countable or otherwise). What they mean, then, by the formula 'the reals are uncountable' is something totally different from what we mean by the same formula. We have different thoughts and that is why both can be true. Here is a summary of Putnam's argument.

1. We can raise such questions as 'are we brains in a vat? are there elm trees? are the reals uncountable?' These are all legitimate questions in our language. We can raise them simply because of our mastery of our language.
2. But no brain in a vat could raise such questions. The reason for this is that, according to Putnam, most words have their meaning solely in virtue of referring to what they do. Meanings are not mental or brain states: 'meanings just ain't in the head', as he puts it. So to ask a question about elm trees or the real numbers, one must be able to refer to them. But this is just what the brains in a vat can't do. When constructing that scenario, we left the elm trees and most of the reals out of the vat! If they ask meaningful questions, it must be in reference to items in their vat universe.
3. It follows, therefore, that we are not brains in a vat, countable or otherwise.

Putnam's argument about the general brains in a vat case strikes many people as preposterous. I have defended it elsewhere (see note 9), so I won't repeat that defense here. What I would like to emphasize is how much more plausible it is when deployed against the Skolem Paradox. After all, there is something crazy about the suggestion that we are living in a countable model. We can prove that the reals are uncountable. That proof gives meaning to that term. Ultimately - and this is Putnam's deep point - it doesn't make sense to wonder whether the reals are 'really countable' according to some hypothetical, alien concept of 'countability' to which we, by hypothesis, have no access. We are stuck with our words in the universe we inhabit. The very idea that we are in a countable universe is semantic nonsense, a grammatical illusion. For in that idea, the words 'countable' and 'universe' have been divorced from their usual meaning (according to which the question is easily answered 'no'). The words are now supposed to get their meaning somewhere else, from the outsiders' point of view. But from our point of view, that question - 'are we living in a countable universe' as part of an alien language - is no more meaningful

than - 'are we living in a 'gffflh psst?'

To put the point another way, the countable model exists as a possibility only relative to our world. It is defined to be different from our world, an interpretation alternative to the standard (ie., original) one we begin with. We construct the alternative by leaving certain things out of the model (eg., almost all the real numbers). It is totally incoherent to turn around and wonder whether *that* world might not be ours after all. Moreover, this argument is not one bit weakened by the fact that mathematicians in a countable world can give it, for it expresses a deep point about reality. The point is that the distinction between countable and uncountable sets is an internal distinction, a distinction within our mathematical theory. (What else could it be?) It makes no sense to wonder whether sets really are countable in some absolute sense, apart from the sense we, following Cantor, give this term in the context of our mathematics. (Exactly the same kinds of reasons can be given by Putnam when showing that we are not brains in an ordinary vat.)

In both cases, Putnam's arguments turn on a philosophy of language, an account that stresses the role of use or practice in determining meaning. For Putnam, there are no mental meanings that mediate reference. Our worlds, like '3', 'pi', 'omega', 'reals', achieve their semantic import in terms of the references they make. In this sense, Putnam's philosophy of language is very realist; reference is the basic semantic relation. But reference is not a magical or puffed up relation. In the end, it is explained by the simple schema that referring term 't' refers to t. To say that words essentially refer is not to posit some magical relation between words and a collection of objects that exists independently of us. Putnam calls his view 'internal realism' for the reference relation is construed as an internal relation between two different aspects of our conceptual scheme, words and objects.

Thus Putnam's internal realism does not commit set theory to the unbridled realism that Maddy defended in the article mentioned in the beginning of this essay. To say that there are sets, infinite, even uncountable sets, is just to say that some set theoretical terms are referential. It does not follow from this that there is a full and determinate set theoretic reality such that every proposed axiom is either true or false of it independently of our conceptions. Rather, in the ideal, what is true is simply what is rational for us to accept¹⁵ (recall Thom, 'a proof is what convinces mathematicians').

So the Putnam solution to the Skolem puzzle is, like the quasi-empiricism I defended in *New Directions*, neutral with

respect to the degree of realism we espouse. But what of all that talk of informal mathematics and human nature that appeared in Benacerraf, Wright and (by implication) Wittgenstein? If it doesn't solve this version of the Skolem paradox, then what is its point in discussing set theory?

Wittgenstein was quite adamant about what he thought were conceptual confusions that surrounded set theory. 'Infinite sets are not large sets' he warned and he claimed that set theory made the fundamental error of treating finite collections and infinite series (or laws) with the same notation, a notation for arbitrary sets.¹⁶ The mistake, Wittgenstein said, was to read into the technical notions of set theory, 1-1 mappings, countability, uncountability, and so forth, analogies with considerations of size about finite domains. Indeed, Wright goes on in the above mentioned article to raise similar doubts. Perhaps the countable-uncountable distinction has nothing to do with size, but only with the relative complexity of certain sets. (Notice how close this answer comes to the purely formalist reply. All there are are the technical mathematical notions which might as well be called C-able and unC-able. It is mistake, according to Wittgenstein, to color these notions with shades borrowed from the finite domain. It is sullyng mathematics with unnecessary and harmful 'prose'.) Notice, too, that at this point Wittgenstein is interfering with the mathematicians in just the way that he claimed that philosophy had no right to do. 'But I'm not interfering with their mathematics' he would claim, 'just with their prose. I don't want to drive them out of paradise but only to show them set theory is not a paradise and then, they'll leave of their own accord'.

Most mathematicians, I feel sure, will reject these Wittgensteinian objections. Indeed, as Shanker makes clear (unintentionally) the objections are unconvincing and often reduce to a sort of moral outrage at the obstinacy of mathematicians.¹⁷ But I propose a novel way of saving Wittgenstein from himself. Suppose that the grounding of our concepts, our human nature, or natural inclinations, what Wright calls sub-rational faculties and natural propensities, include making the very analogies that Wittgenstein inveighs against. That is, what if what Wittgenstein regarded as the prose, the color, is part of the mathematics and can not be separated from it? We are creatures that are naturally inclined to see sets as including infinite and finite sets. This needs no justification, this is just how we act. So there is more to the concepts of set theory than are provided by any formal system. But that more is provided by our own natures, not by any external reality. In other words, I'm suggesting that it matters what we *call* our mathematical concepts. The term

'large cardinals' inspires and guides mathematical activity in a way that 'curious and intricate set theoretic constructions' cannot. Mathematicians prove things in set theory by imagining the transfinite universe, by taking their prose seriously, just as science fiction writers solve their problems by taking their imaginary universes seriously. In neither case, do we have anything beyond the practices and expert practitioners. In both cases the practices give meaning that no formal specification can exhaust because the meaning includes the color, the direction and the excitement of the practices.

If these last remarks are at all correct, then we have a novel account of Wittgenstein's own hostility to the infinite. He becomes a perfect example of someone drummed out of the mathematical community for failing to catch on! As his view allows and he himself admitted, there can be people who don't share the required inclinations of mathematicians - and Wittgenstein doesn't. He describes the way a tribe teaches its children how to count as follows.

The children of the tribe learn the numerals in this way: They are taught the signs from 1 to 20... and to count rows of beads of no more than 20 on being ordered, "Count these". When in counting the pupil arrives at the numeral 20, one makes a gesture suggestive of "Go on", upon which the child says (in most cases at any rate) "21"... If the child does not respond to the suggestive gesture, it is separated from the others and treated as a lunatic.¹⁸

When learning set theory, mathematicians are presented finite sets of increasing size, and finally infinite sets and are told the latter are very much larger than the largest finite sets presented. Eventually they are shown Cantor's Diagonal Argument and told that this shows the reals are vastly larger than anything seen so far. They are encouraged to name other large sets. If a mathematician can not do this, refuses to do this, complains about unwarranted transference from the finite domain to the infinite, then he is separated from the tribe of mathematicians and treated as a lunatic. Well, perhaps not as a lunatic, but certainly not as a mathematician. This is, I suggest, what happened to Wittgenstein at least with regard to set theory. He could see alternatives where others could not; he could see what held mathematical practice together was neither insight into a platonic realm nor complete specification of a formal system, but nothing more than human nature, the shared inclinations, the natural properties, the sub-rational nature of human mathemati-

cians. But at a crucial point, his own inclinations led him astray. He is, to use his own term, a lunatic, not because his views on infinite sets are crazy, but because they are not shared by his community.

In conclusion then, it seems that mathematical practice, not scientific methodology, is what really matters to quasi-empiricist approaches, at least in set theory. If we are to take a realist attitude at all, then only Putnam's form of internal realism can save us from Skolem skepticism. However, the realism that emerges does not justify a quasi-scientific attitude in set theory. There is no transfinite world out there, complete in all its details, waiting to be discovered. When we learn to accept the notion of an arbitrary set of integers, our acceptance is not grounded in a transfinite world; we are rather like readers who learn to accept the idea of time travel. In both cases, acceptance amounts only to continuing on with the story. The realism of set theory is a feature of mathematical practice, of the way we talk about sets, of the metaphors we use and the analogies we draw, of our mathematical prose and the proofs we choose to puff up. Wittgenstein was right to notice the puffed up proofs and the colourful prose of set theoreticians. He was wrong to think that set theory, or any mathematics, can be done without such color. Set theory is not, as Maddy suggests, a quasi-science. It is a quasi-art.

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NOTES

1. *New Directions in the Philosophy of Mathematics*, Boston: Birkhäuser, 1986.
2. For a discussion of the Four Color proof, see my essay in *New Directions*. For a report on the proof of the non-existence of a finite projective plane of order 10, see *The New York Times*, Dec. 20, 1988, pp. C5, C17.
3. *Ibid.*
4. "Believing the Axioms." Parts I and II. *The Journal of Symbolic Logic*, Vol. 53, Nos. 2,3 (1988): 481-511, 736-764.
5. "Modern Mathematics: An Educational and Philosophic Error?", reprinted in *New Directions*.
6. See, for example, his *Remarks on the Foundations of Mathematics*, Oxford: Blackwell, 1956, and *Wittgenstein's Lectures on the Foundations of Mathematics*, 1939, C. Diamond (ed.), Sussex

7. Descartes, *Meditations on First Philosophy*, the First Meditation. The quotation is from the translation in *Descartes, Philosophical Writings*, E. Anscombe and P. Geach (eds.), London: Nelson, 1969.
8. *Word and Object*, Cambridge: MIT Press, 1960, especially Chapter One. Also "Epistemology Naturalized" in *Ontological Relativity and Other Essays*, New York: Columbia University Press, 1969. Despite his conclusions, Quine is not a skeptic because, as he says, he takes the original starting theory seriously.
9. The next two paragraphs are taken from my essay "In Defense of Putnam's Brains"; *Philosophical Studies*, forthcoming.
10. The first indication of such relativism is in Skolem's 1922 paper "Some Remarks on Axiomatized Set Theory". For a full reference to this and to other of Skolem's views, see Paul Benacerraf's fine article, "Skolem and the Skeptic", *The Aristotelian Society, Supplementary Volume LIX, 1985*, pp. 85-116
11. *Lectures on the Foundations of Mathematics*, 103.
12. Benacerraf, op. cit.
13. "Skolem and the Skeptic", loc.cit., pp. 117-137.
14. *Reason, Truth and History*, Cambridge: Cambridge University Press, 1981, especially the first three chapters. The connection between Putnam's argument about brains in a vat and the modified Skolem paradox is a major focus of the paper mentioned in note 9 and is briefly discussed by Benacerraf.
15. *Reason, Truth and History*, preface.
16. For a vigorous defense of Wittgenstein's position on the infinite, see Stuart Shanker's *Wittgenstein and the Turning Point in the Philosophy of Mathematics*, Albany: SUNY Press, 1987, chapter 5.
17. Ibid.
18. *The Blue and Brown Books*, Oxford, Blackwell, 1960, p. 93.