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ON THE INCOMPLETENESS OF AXIOMATIZED MODELS FOR THE EMPIRICAL SCIENCES*

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1. *The starting point*

Chaos theory has been a fast-growing research area since the early 70's, a decade after the discovery of (an apparently) chaotic behavior in a deterministic nonlinear dynamical system by E. Lorenz (for references see [4]). Chaos scientists usually proceed in one of two ways: whenever they wish to know if a given physical process is chaotic the usual starting point is to write down the equations that describe the process and out of them formally check whether the process satisfies some of the established mathematical criteria for chaos and randomness. However those equations are in most cases intractable nonlinear differential equations; moreover, in general they have no analytical solutions. Therefore chaos theorists turn to computer simulations. Usually a Mac or a PC will do the trick: the simulation is easily done and for most nonlinear systems one sees a confusing, tangled pattern of trajectories on the screen.

The system *looks* random, chaotic. Better: there are statistical tests such as the Grassberger-Proccacia criterion that guarantee the existence of randomness in a computer-simulated system given certain presuppositions and within a margin of error. Yet statistical tests furnish no *mathematical* proof of the existence of chaos in a dynamical system. There is always the chance that the system is undergoing a very long and complicated transient state, before it settles down to some nice and regular behavior. Therefore how can we prove that a dynamical system that *looks*

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chaotic is, in fact, chaotic?

That problem has been around for some time since the discovery and early exploration of what is now called "deterministic chaos". Researchers in the area either try to explore (as we have just explained) through computer simulations well-known systems that can be mathematically described, to see if they have the looks of a random system, or try to develop finely-tuned formal criteria for chaos that can be mathematically checked in a system, not just inferred out of a disordered appearance of the system's trajectories.

In a 1983 conference (published in 1985) Morris Hirsch stated that time was ripe for a marriage between the "experimental" and "theoretical" sides of chaos research: after discussing the Lorenz equations, Hirsch remarks [20]:

> (...) By computer simulation Lorenz found that trajectories seem to wander back and forth between two particular stationary states, in a random, unpredictable way. Trajectories which start out very close together eventually diverge, with no relationship between long run behaviors.
>
> But this type of chaotic behavior has not been proved. As far as I am aware, practically nothing has been proved about this particular system (...).
>
> A major challenge to mathematicians is to determine which dynamical systems are chaotic and which are not. Ideally one should be able to tell from the form of the differential equations.

More recently, in a 1990 conference, S. Smale formulated a closely related problem about the Lorenz system and general chaotic dynamical systems [29]:

> Are the dynamics of the Lorenz equations described by the geometric Lorenz attractor of Williams, Guckenheimer and Yorke?
>
> Also, the general problem of establishing and analyzing strange attractors of differential equations of physics and engineering is still wide open.

Smale asks for a proof that the Lorenz system has an attractor related to the Williams-Guckenheimer-Yorke (WGY) attractor; he also asks for a general characterization of chaos in "concrete" (or "naturally occurring") dynamical systems. Hirsch asks for a *decision procedure* to test for chaos in a system. If such a procedure were available, given the WGY attractor one would immediately solve Smale's problem for the Lorenz equations just by applying it.

However we showed [4] that no such a decision method exists. Moreover, for any nontrivial characterization of chaos in a dynamical system there will always be systems where proving the existence of chaos is unatainable within reasonable standard axiomatizations. Chaos theory, as well as dynamical systems theory, are both undecidable — there is no general algorithm to test for chaos in an arbitrary dynamical system — and incomplete — there are infinitely many dynamical systems that will look chaotic on a computer screen, for they are chaotic in an adequate class of standard models for axiomatized mathematics, but such that no proof of that fact will be found within the usual formalizations of dynamical systems theory. Similarly, there are systems whose properties are (formally) equivalent to the proof of intractable problems such as Fermat's Conjecture, or Riemann's Hypothesis, or the *P?NP* question [10], and those systems are densely dispersed in a natural topology among all dynamical systems [13]. Classical mathematics is dramatically incomplete in the sense of Gödel, and full of extremely difficult problems, that can arise in innocent-looking contexts.

Worse yet: all our first examples for undecidability and incompleteness within axiomatized physics could be formally reduced to elementary arithmetic problems [4] [13]. However we later discovered that that reduction cannot always be made, as we can obtain examples of intractable problems in the axiomatized sciences which are *not* elementary number-theoretic problems in disguise. They stand beyond the pale of arithmetic; they are much more difficult than any arithmetical problem, and yet they look like commonsensical mathematical statements.

There are even weirder situations: we can obtain formal expressions that describe physical systems such that *nothing* but trivialities can be proved about them. And again those systems may be shown to lie fully outside the arithmetical hierarchy, since they belong to the non-arithmetical portion of set theory (if we are working, say, within Zermelo-Fraenkel set theory). Those are truly faceless systems, very much like generic sets in forcing models; however their construction shows no relation to the usual forcing tools and can be proved to be outside the reach of the usual forcing techniques.

Our results are consequences of general incompleteness theorems that can be found in our papers [4] [5] [6] [7] [8] [9] [13] [14] [15] [16] [17]; other references are [24] [32] [33]. In the present paper we summarize and state without proofs our chief results, with a few comments to clarify their meaning; an Appendix sketches a more formal treatment, which is fully available in the references.

2. *Axiom systems and independence proofs*

Mathematical sentences that are undecidable with respect to sensible axiomatic systems have been known since the 19th century proof of the independence of the Parallel Postulate from the remaining axioms and postulates of geometry. Here we have a meaningful and "intuitively true" assertion in a "natural" model for geometry which couldn't be deduced from the then available axiom system for that discipline. So, nothing new here.

However Euclid's system is notoriously inadequate according to our current criteria for mathematical rigor (even if the 19th century independence results remain when geometry is reformulated in today's language). The main surprise that stemmed from Gödel's 1931 incompleteness theorems is the conclusion that, even if we adhere to contemporary standards in the formulation of mathematical proofs, all the usual axiom systems strong enough for most of mathematics turn out to be incomplete; it is enough that they include arithmetic for undecidable sentences to creep up within them.

Yet, due to Gödel's weird examples of undecidable sentences, the hope remained that undecidability and incompleteness would always be peripheral to mainstream mathematics. That is to say, everyday mathematics, as practiced by the professional mathematician, would be untouched by Gödel's stormy results. (In a recent interview René Thom expressed that same hope, when he said that Gödel's results were to be seen as "road signs," "warning posts," meaning that one shouldn't go further in that direction, but that they had no meaning for the practicing mathematician [37].)

Cohen's independence proof of the continuum hypothesis from the axioms of Zermelo-Fraenkel set theory shattered that hope, since the continuum hypothesis affects innumerable important results in topology, analysis and even algebra.[1] As it is well known, forcing techniques led to the independence proof of several open questions in mathematics such as Whitehead's problem in the theory of abelian groups, but the application of forcing demands high mathematical ingenuity, while being restricted to nonabsolute assertions in set theory. Thus, finite objects — which are absolute due to their finiteness — remained outside the scope of forcing, while the success of forcing applications and techniques strengthened the feeling that the domain of the finite should be regarded as the ground plan from which arose the whole of mathematics.

That definitely seems today to be a rather risky assumption.

3. *Hilbert's 6th and 10th problems*

Around 1987 the present authors started a research program whose main goal was to fully axiomatize physics and to apply modern techniques from mathematical logic to problems in physics. The motivation was found in a mathematical landmark: the 1900 list of 23 problems that David Hilbert presented to the Second International Congress of Mathematicians in Paris. The sixth problem in Hilbert's list asks for an axiomatic formulation of physics; the tenth problem asks for a decision procedure to verify whether a polynomial Diophantine equation with integral coefficients does have solutions. Both problems are fused in our results.

We had a twofold goal in that program. First, we wished to place physics (and, if possible, any mathematically-formulated area in the other empirical sciences) upon a firm and rigorous footing (according to current conceptions). Second, we wished to obtain "meaningful" undecidable sentences within those theories. Somehow we hoped that formally undecidable assertions in an empirical science might turn out to be, let us say, "empirically" decidable. In order to proceed we drew up a list of a few problems that might lead to our goal. We believed that classification schemes for spacetimes in general relativity might allow the construction of unsolvable problems and of undecidable sentences within the corres-

[1] But recall that the algoritmic unsolvability of Hilbert's 10th Problem as well as the Paris-Harrington result imply examples of sensible undecidable sentences in elementary arithmetic.

ponding theory, and we had the feeling that Hirsch's decision problem for chaos theory [20] was algorithmically unsolvable, and that its investigation might prove fruitful. As our tool we only had forcing at the beginning of our investigation, and forcing was applied with modest results to the first question and related areas [8] [14]. It soon became apparent that we had to develop new methods to attain our goals. The literature on forcing has mainly to do with fine technical points in set theory and in the theory of infinite cardinals but for a few well-described resounding results such as Solovay's on measurable sets over the real line. It was never evident to us how to apply those results to the empirical sciences, and the few consequences we could squeeze out of forcing within some axiomatized theories look quite far from everyday life [8] [14].

4. *Suppes predicates and the axiomatics of classical physics*

Suppes' ideas on the axiomatization of theories with the help of set-theoretic predicates gave us the first essential breakthrough in our treatment of decision problems in physics. The whole road from the way physics is practised by a theoretician towards the axiomatic treatment we developed for it is pretty spontaneous. Consider the example of classical mechanics. A rigorous and all-encompassing treatment of mechanics was first given by C. Lanczos in his wonderful treatise on *The Variational Principles of Mechanics* in 1949 [23]. Lanczos, who was very close to A. Einstein, inspired himself in the earlier, brilliant treatise by Hertz [19], and formulated classical mechanics as an interpretation of some geometrical structures from Riemannian geometry. About ten years later R. Palais circulated a set of notes where classical mechanics was garbed in a new geometrical dress, the language of fiber bundles, connection forms and symplectic geometry. The first time that new presentation of classical mechanics appeared in book form was as a chapter in S. Sternberg's textbook, *Lectures on Differential Geometry* [31]; we can also quote S. MacLane's notes [26]. However the definitive presentation of classical mechanics as an interpretation of the differential geometry of symplectic bundles and related structures is to be found in the Abraham-Marsden treatise, *Foundations of Mechanics*, whose first edition appeared in 1967 [1].

Our axiomatic treatment for mechanics stems from that lineage. Once we have translated mechanical concepts as geometrical structures within an adequately rigorous mathematical context, we can reformulate everything inside the framework of Zermelo-Fraenkel set theory and summarize the whole of mechanics as a single (albeit very complicated) Suppes-like set-theoretic predicate within a first-order language [3] [7] [8] [9].

An axiomatic treatment adds a tremendous amount of extra rigor to our presentation of an empirical science. Yet the added rigor is meaningless if we can't pile up something new on it. As we said, we started looking for independence results in the axiomatized empirical sciences, since independence results are the trademark of axiomatic systems framed in classical first-order languages that include arithmetic. The path that led to forcing was cumbersome and unpromising; again P. Suppes helped us when he suggested that we should check Richardson's [27] 1968 examples of unsolvable problems in analysis and "see if they have any application in quantum mechanics, as they deal with sines and cosines." (We quote Suppes' words [36].)

However we immediately noticed that Richardson's constructions were in fact realizations of a functor from the theory of formal systems (here coded as Diophantine equations) into classical elementary analysis. We had some previous intuition that such a fully algorithmic functor might exist, one that would translate metamathematics into questions about elementary functions and their properties. Yet we had no example of such a functor before we learned of Richardson's results [11] [12].

Richardson's results are framed as *undecidability* results; as such they look rather weak, and it isn't immediately apparent that they in fact imply a full-fledged *incompleteness* theorem for the language of classical analysis. (Kreisel [22] had previously delved on them, while Suppes was at first rather skeptical about the incompleteness phenomena implied by Richardson's examples.)

5. *Incompleteness*

Out of Richardson's examples we immediately obtained almost by chance something that had previously seemed impossible, an expression for the halting function within a rather simple mathematical language, the language of classical analysis: let $M_n(q)$ be the Turing machine of index n that acts upon the natural number q [28]. Granted that ϕ (n, q) be the *halting function* for $M_n(q)$, ϕ (n, q) = 1 if and only if $M_n(q)$ stops over q, and ϕ (n, q) = 0 if and only if $M_n(q)$ doesn't stop over q.

We suppose that our theories T are *arithmetically consistent*, that is, that the model we are interested in represents arithmetical assertions by the standard model for arithmetic. (One might take T $\supseteq$ ZFC.) Let $p_{n,q}(x_1, x_2, \ldots, x_n)$ be a universal Diophantine polynomial [21]. Let σ be the sign function, $\sigma(\pm x) = \pm 1$, $\sigma(0) = 0$. Then:

Proposition 5.1 (The Halting Function.) *If T is arithmetically consistent, then*:

$$\phi(n, q) = \sigma(G_{n,q}),$$

$$G_{n,q} = \int_{-\infty}^{+\infty} C_{n,q}(x) e^{-x^2} / (1 + C_{n,q}(x)) dx,$$

$$C_{n,q}(x) = \lambda\, p_{n,q}(x_1, \ldots, x_r). \quad \square$$

(λ is one of Richardson's maps from the Diophantine polynomials into elementary real analysis [4].)

Out of that we proved a first general undecidability and incompleteness theorem: we say that a predicate P in our formal language is *nontrivial* if there are term-expressions ξ, ζ in our theory T such that $T \vdash P(\xi)$ and $T \vdash \neg P(\zeta)$.

Then:

1. In T, given any nontrivial predicate P there is a countably infinite family of term-expressions ξ_m such that there is no general algorithm to decide, for an arbitrary m, whether or not $P(\xi_m)$.
2. Even if we can prove that $T \vdash P(\xi_m)$, the function g(m) that bounds those proofs (whenever they can be done) isn't recursive. Therefore those proofs may be arbitrarily difficult.
3. There are (denumerably) infinite many term-expressions ξ in our language such that in an adequate model **M** it is true that $P(\xi)$, while our theory T neither proves nor disproves that assertion.

Things go much farther, and finiteness is no hindrance here; as an example we now quote a recent result on the incompleteness of the theory of *finite* noncooperative games with Nash equilibria, a result that has immediate relevance for neoclassical economics [16].

(As it is well-known, the theory of games was developed by John von

Neumann out of some ideas and a major conjecture by Emile Borel with a view towards its applications in economics; in the early 50's Kenneth Arrow and Gérard Debreu translated the theory of competitive markets into the language of game theory, which allowed them to prove the central result of Walrasian neoclassical economics: every competitive market has a set of equilibrium prices. So, when one talks about games, one is talking about competitive markets.)

Proposition 5.2 *If T is arithmetically consistent then*:

1. Given any nontrivial property P of finite noncooperative games, there is an infinite denumerable family of finite games Γ_m such that for those m with $T \vdash$ "$P(\Gamma_m)$," for an arbitrary total recursive function $g:\omega \to \omega$, there is an infinite number of values for m such that the length of the proof of $P\Gamma_m$ from the axioms of T is strictly larger than $g(\|P\Gamma_m\|)$, where $\|P\Gamma_m\|$ is the length of the formal expression that describes $P\Gamma_m$ in the language of T.
2. Given any nontrivial property P of finite noncooperative games, there is one of those games Γ such that $T \vdash$ "$P(\Gamma)$" if and only if $T \vdash$ "Fermat's Conjecture."
3. There is a noncooperative game Γ where each strategy set S_i is finite but such that we cannot compute its Nash equilibria.
4. There is a noncooperative game Γ where each strategy set S_i is finite and such that the computation of its equilibria is T-arithmetically expressible as a Π_{m+1} problem, but not to any Σ_k problem, $k \leq m$.
5. There is a noncooperative game Γ where each strategy set S_i is finite and such that the computation of its equilibria isn't arithmetically expressible.

Here lies the big surprise: *everything* turns out to be undecidable; each nontrivial property, even the simplest one, leads to an incompleteness proof. There are natural problems that turn out to be as difficult as Fermat's problem; there are natural problems that are equivalent to arithmetic problems as high as one wishes in the arithmetical hierarchy; and there are natural problems that lie outside the arithmetical hierarchy.

Our constructions essentially arise from the existence of expressions for the halting function and for characteristic functions in all arithmetic degrees of unsolvability within elementary analysis. No forcing is required; in fact we still wonder why those results weren't discovered earlier. The main constructions are pretty straightforward; however in order to obtain them we had to believe from the very beginning that incompleteness is something that belongs to the way we conceive mathematics. Incompleteness is a natural phenomenon according to our current views about mathematics. And yet everybody seemed to shy away from that fact of mathematical life.

We can give an idea of the scaffolding that supports our results. Consider a universal Turing machine. Assuredly we can recursively enumerate the natural numbers over which the machine stops, but the complementary set of numbers which give rise to never-ending infinite loops forms, as we know, a productive, non-recursively enumerable set. Suppose now that we have reconstructed Turing machine theory within elementary arithmetic A. We have a list of theorems like A⊢ "The universal machine U(**n**) stops," where **n** is a natural number such that, in fact, U(**n**) stops. However the listing of theorems of the form A⊢ "The universal machine U(n) never stops" cannot exhaust all possibilities for nonstopping machines, and so there will be a **n'** such that U(**n'**) never stops (in the 'real' world and in a standard model), but A isn't strong enough to prove it.

That result is the essence of Gödel's first incompleteness theorem. It can also be given the following interpretation: within our formal theory A, in the place of a universal machine U we can take an universal polynomial p over the integers **Z** parametrized by **n** [21]. Sentences such as "The universal machine U(**n**) never stops" become equivalent to "The polynomial p(**n**, ...) has no roots over **Z**." And if A cannot prove that U will never stop over **n'**, it will never be able to prove that p(**n'**,...) = 0 has no integer solutions. With the help of Richardson's functor we then translate those ideas into the domain of smooth and piecewise smooth elementary functions in analysis.[2]

6. *More results*

Our undecidability and incompleteness theorems have proved extremely fruitful. The following problems have been dealt with our methods:

[2] For technical details see the Appendix.

- *The integrability problem in classical mechanics.* There is no general algorithm to decide, for a given hamiltonian, whether or not it is integrable. Also there will be both integrable and nonintegrable hamiltonians in **M** but such that T is unable to prove it [4].
- *The Hirsch problem.* Is there an algorithm to check for chaos given the expressions of a dynamical system? No: there is no such a general algorithm, and there will be systems that look chaotic on a computer screen (that is to say, they are chaotic in our model **M**) but such that proving their chaotic behavior is impossible in T [4] [20].
- *Penrose's thesis.* Penrose conjectured that classical physics offers no examples of noncomputable phenomena. We gave a counterexample to that assertion [5] [32] [33].
- *"Smooth" problems equivalent to hard number-theoretic problems.* We gave an explicit example of a dynamical system where there will be chaos if and only if Fermat's conjecture is provable. We also showed that (given some conditions) those 'nasty' problems are dense in the space of all dynamical systems [13].
- *Arnol'd's problems.* Arnol'd formulated in the 1974 AMS Symposium on the Hilbert Problems [2] some questions dealing with algorithmic decision procedures for polynomial dynamical systems over **Z**. We showed that again there are no general algorithms available, and that the theory of those systems is incomplete [9] [15].
- *Problems in mathematical economics.* Lewis [24] pointed out that our results entail the incompleteness of the theory of hamiltonian models in economics. They also entail the incompleteness of the theory of Arrow-Debreu equilibria and (what is at first sight surprising) the incompleteness of the theory of *finite* games with Nash equilibria [9] [16].
- *Problems worse than any number-theoretic problem.* They can be constructed (and look "natural") with our techniques [9].

7. *Forcing and our techniques*

Which is the relation between our techniques for the construction of undecidable statements and the Cohen-Solovay kind of forcing? In order to answer that question we must conceive a theory as a Turing machine that accepts strings of symbols — well-formed formulae — and stops whenever those strings of symbols are theorems of the theory. If not, it never halts and enters an infinite loop.

Now consider Zermelo-Fraenkel axiomatic set theory, ZF. If M_{ZF} is the corresponding proof machine for ZF, and if CH is the Continuum Hypothesis, we know that $M_{ZF}(CH)$ never halts. Accordingly, there is a Diophantine polynomial $p_{ZF}(CH, x_1, \ldots)$ that has no roots over **Z**, but since CH is independent of the axioms of ZF, there can be no proof (within ZF) of the statement "$p_{ZF}(CH, x_1, \ldots) = 0$ has no roots over **Z**." (If there were one such proof, we would then be able to decide CH in ZF.) With the help of our techniques (see Proposition 5.1 above) we can obtain a two-step function $\phi_{ZF}(m)$ such that, if m_{CH} is a Gödel number for CH, then both ZF $\nvdash \phi_{ZF}(m_{CH}) = 0$ and ZF $\nvdash \phi_{ZF}(m_{CH}) = 1$. Therefore, every undecidable statement constructed with the help of forcing within ZF (or even within weaker theories, provided that they include elementary arithmetic) gives rise to undecidable statements according to the present tools; if the theory considered is too weak to encompass analysis, then we can apply to it a construction like the one in Proposition A.18 in the Appendix with the same results.

Moreover the converse isn't true, that is, there are some (actually, infinitely many) undecidable statements which can be constructed according to the present techniques, but such that no forcing statement will be mapped on them if we follow the preceding procedure. Finite objects are forcing-absolute, but we have seen that we can construct undecidable statements about finite objects in ZF say, again through the ϕ function in Proposition 5.1. If m_{Fin} is the Gödel-coding for one of those statements, then "$\phi_{ZF}(m_{Fin}) = 0$" cannot be proved in ZF. So, there is a (metamathematical) algorithmic procedure that goes from every undecidable statement in ZF onto undecidable statements about the ϕ_{ZF} function; and yet forcing statements are only a portion of that map, since there is much more in it.

How general are our results? Are there undecidable statements beyond them?

First, we must add that it isn't clear whether there are other techniques besides Cohen-forcing for the construction of undecidable statements within fragments of set theory. We believe however that one will eventually *prove* that there are infinitely many such particular techniques, which will turn out to be irreducible to forcing. (At present, that assertion is just a matter of faith in the inner wealth of mathematics.) Second, our technique is a truly general one; it starts out of any previously proved undecidable statement within an axiomatic system and allows the construction of infinitely many new undecidable statements of any desired degree of difficulty within the arithmetic hierarchy and even beyond. General results tend to be very simple, and so are our main ideas. They are also, as we have seen, highly effective. A point to be made is that the semantics for those undecidable statements of us require nonstandard models: all Diophantine equations p(...) = 0 behind our undecidable statements have no roots in standard models but have integer solutions in nonstandard models. Now if we start from two contradictory, forcing-induced undecidable statements ξ and $\neg\xi$ with respect to ZF supposed consistent, the forcing-dependent semantics for both ZF + ξ and ZF + $\neg\xi$ is given by *standard* models; the transformed theories ZF + ξ + $\phi_\xi = 0$ and ZF + $\neg\xi$ + $\phi\neg_\xi = 0$ have also standard models, but we require *nonstandard* models for the new ZF + ξ + $\phi_\xi = 1$ and ZF + $\neg\xi$ + $\phi_{\neg\xi} = 1$. The epistemological consequences of such a multiplication of models are still unclear.

Which is the weakest axiomatized theory to which our constructions can be applied? There is a beautiful theorem by S. Feferman ([28], p. 171ff) where one learns that, given any recursively enumerable Turing degree $a \leq_T \mathbf{0}'$, we can algorithmically construct an axiomatized theory T of that degree. Again there is a Turing machine M_T that stops over inputs coding well-formed formulae that are theorems of T, and otherwise never halting. We can repeat the argument sketched above if we can write down polynomials in T and if we can at least obtain an expression as the one in Proposition A.18. Examples of theories where those constructions are possible can be found in [30], p. 334ff. However it isn't clear whether there are similar constructions for *every* weak theory. Anyway our construction can be easily done for all theories of Turing degree $\mathbf{0}'$ and is perfectly general for theories at that level of unsolvability.

8. *A look beyond*

Let's reduce our ideas and techniques to their bare essentials. Suppose that we start from classical axiomatic set theory, which is a framework big enough to contain all of everyday mathematics within it. Let's look at set theory as an abstract construction, a collection of strings of symbols from an alphabet — the collection of all sentences of set theory. We know that we have here a recursively enumerable sequence of objects which can be effectively mapped onto the natural number sequence, so that we can get the whole of our axiomatic sequence coded (through a Gödel numbering) into an infinite recursively enumerable subset of the natural number sequence. Such a sequence an be embedded into several *continuous* mathematical structures within set theory itself! One of those maps is Richardson's, which we have just started to exploit, but there are infinitely many others.

Once we have thus coded the whole of mathematics into itself, a whole new family of questions appears: say, since we have mapped an axiomatic system into a much larger structure, can we now go back and define out of our embedding some "hyperaxiomatic" structure with brand-new (and sensible) truths and theorems? Do we get something really new here, or can we reduce our hyper-extensions to the traditional setting of first-order recursively enumerable theories?

We know the answer: there definitely are several new results to be found in those maps of mathematics redrawn over its own belly.

9. *Conclusion*

However we would like to emphasize a more modest and yet very important point here. We do it by repeating the conclusion in the first of our papers [4]:

> What can we make out of all [those manifold incompleteness theorems]? We cautiously suggest that the trouble may lie not in some essential inner weakness or flaw of mathematical reasoning, but in a too narrow, too limited concept of formal system and of mathematical proof. There is a strongly mechanical, machinery-like archetype behind our current formalizations for the idea of algorithmicity that seems to stem from an outdated 17th century vision *à la* Descartes (even if our current notion of proof is traced back to Greek mathematics). Also a first-order language such as the one for Zermelo-Fraenkel theory is too weak: even if we can prove all of classical mathematics within it, it is marred by the plethora of undecidability and incompleteness results that we can prove about it, and which affect interesting questions that are also relevant for mathematically-

based theories such as physics.

The authors certainly do not know how to, let us say, safely go beyond the limits of the presently available concepts of computability, algorithmicity, and formal system, but they feel that if there are so many quite commonplace things that 'should' somehow be provable or decidable within a sensible mathematical structure, and which however turn out to be algorithmically undecidable or unprovable, then one cannot blame the whole of mathematics for that. Mathematics isn't at fault here. The problem lies in our current ideas about formalized mathematics. They are too weak. We must look beyond them.

10. *Acknowledgments*

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A. *Appendix: The formal machinery*

We suppose that our theories are formalized within a first-order classical language with equality and the description operator.

We follow the notation of [4]; ω denotes the set of natural numbers, **Z** is the set of integers, and **R** are the real numbers. Let T be a first-order consistent axiomatic theory that contains formalized arithmetic **N** and such that T is strong enough to include the concept of set and classical elementary analysis. (We can simply take T = ZFC, where ZFC is Zermelo-Fraenkel set theory with the axiom of choice.) If L_T is the formal language of T, we suppose that we can form within T a recursive coding for L_T so that it becomes a set L_T of formal expressions in an adequate interpretation of T. Objects in T will be noted by lower case letters x, y, x_i, y_i. Predicates in T will be noted P, Q,....

From time to time we will play with the distinction between an object and the expression in L_T that represents it. If x,y are objects in the theory, ξ, $\zeta \in L_T$ are term-expressions for those objects in the formal language of T. In general there is no 1-1 correspondence between objects and expressions; thus we may have different expressions for the same functions: 'cos $(1/2)\pi$' and '0' are both expressions for the constant function 0. We note by $\ulcorner x \urcorner$ an expression for x in L_T. We allow the following abuse of language: predicates P sometimes apply to objects in T and sometimes apply to expressions in L_T (say, P(x) or P(ξ)); meaning will be clear from context.

We emphasize that proofs in T are algorithmically defined ways of handling the objects of L_T; for the concept of algorithm see [4] [28]. A review of concepts from computation theory and applications (algorithms, Turing machines, formal systems and the like) can be found in [4] [25] [28]. Ideas from logical number theory, such as the Matijasevich-Davis-Robinson-Jones theorem and universal polynomials can be found in [18] [21].

Undecidability and Incompleteness in T

Definition A.1 T is **arithmetically consistent** if and only if the standard model N for **N** is a model for the arithmetic sentences of T [4] [13]. □

Let now $\mathcal{P}$ be the algebra of polynomial expressions on a finite number of unknowns over the integers **Z**; we identify $\mathcal{P}$ to the set of expressions for Diophantine polynomials in T. Let ε be the set of expressions for real elementary functions on a finite number of unknowns, while $\mathcal{F}$ is the set of expressions for real-valued elementary functions on a single variable.

We assert [4]:

Proposition A.2 (Richardson's Functor.) Let $p_m(x_1,x_2,\ldots,x_n) = 0$ be a family of expressions for Diophantine equations parametrized by the positive integer m in an arithmetically consistent theory T. Then there is an algorithmic procedure $\alpha : \mathcal{P} \to \varepsilon$ such that out of $p_m \in \mathcal{P}$ we can explicitly obtain an expression

$$f_m(x_1,x_2,\ldots,x_n) = \alpha p_m(x_1,x_2,\ldots,x_n),$$

$f_m \in \varepsilon$, such that $f_m = 0$ if and only if $f_m \leq 1$ if and only if there are positive integers $x_1, x_2, \ldots, x_n$ such that $p_m(x_1, \ldots, x_n) = 0$.

Moreover there are algorithmic procedures $\iota', \iota'' : \mathcal{P} \to \mathcal{F}$ such that we can obtain out of an expression p_m two other expressions for one-variable functions, $g_m(x) = \iota' p_m(x_1,\ldots)$ and $h_m(x) = \iota'' p_m(x_1,\ldots)$ such that there are positive integers $x_1,\ldots$ with $p_m(x_1,\ldots) = 0$ if and only if $g_m(x) = 0$ and $h_m(x) \leq 1$, for all real-valued x. □

Proposition A.3 (Incompleteness of Real Analysis.). If T is arithmetically consistent, and if we add the absolute value function $|x|$ to $\mathcal{F}$ and close it to obtain an extended set of expressions $\mathcal{F}^*$ we have:

1. We can algorithmically construct in T a denumerable family of expressions for real-valued, positive-definite functions $k_m(x) \geq 0$ so that there is no general algorithm to decide whether one has, for all real x, $k_m(x) = 0$.
2. For a model **M** such that T becomes arithmetically consistent, there is an expression for a real-valued function k(x) such that $\mathbf{M} \models \forall x \in \mathbf{R}\; k(x) = 0$ while $T \nvdash \forall x \in \mathbf{R}\; k(x) = 0$ and $T \nvdash \exists x \in \mathbf{R} k(x) \neq 0$. □

If k_m (as in Proposition A.3) results out of p_m, we write $k_m = \lambda\, p_m$.

Equality is undecidable in L_T

Corollary A.4 If T is arithmetically consistent then for an arbitrary real-defined and real-valued function f there is an expression $\xi \in L_T$ such that $\mathbf{M} \models \xi = f$, while $T \nvdash \xi = f$ and $T \nvdash \neg(\xi = f)$.

Proof: Put $\xi = f + k(x)$, for k(x) as in Proposition A.3. □

The Halting Function and expressions for complete degrees in the arithmetical hierarchy

Let $p_{n,q}(x_1, x_2, ..., x_n)$ be a universal polynomial [21]. Since $\mathcal{F}$ has an expression for $|x|$ (informally one might have $|x| = +\sqrt{x^2}$), it has an expression for the sign function $\sigma(\pm x) = \pm 0$, $\sigma(0) = 0$. Therefore we can algorithmically build within the language of analysis (where we can express quotients and integrations) an expression for the halting function $\phi(n, q)$ [4]:

Proposition A.5 (The Halting Function.) If T is arithmetically consistent, then:

$$\phi(n, q) = \sigma(G_{n,q}),$$

$$G_{n,q} = \int_{-\infty}^{+\infty} C_{n,q}(x) e^{-x^2} / (1 + C_{n,q}(x)) dx,$$

$$C_{n,q}(x) = \lambda\, p_{n,q}(x_1, ..., x_r). \square$$

Remark A.6 There are infinitely many expressions for $\phi(n, q)$ in L_T; however due to incompleteness some of them will never be proved to equal the halting function in T. □

Definition A.7 A predicate P in L_T is **nontrivial** if there are x, y such that $T \vdash P(x)$ and $T \vdash \neg P(y)$. □

If $\xi \in L_T$ is any expression in that language, we write $\|\xi\|$ for its complexity, as measured by the number of letters from T's alphabet in ξ. Also we define the *complexity of a proof* $C_T(\xi)$ of ξ in L_T to be the minimum length that a deduction of ξ from the axioms of T can have, as measured by the total number of letters in the expressions that belong to the proof. Let P be any nontrivial predicate, and let $B \supset \mathcal{F}$.

Then:

Proposition A.8 If T is arithmetically consistent, then:

1. There is an expression $\xi \in B$ so that $T \nvdash \neg P(\xi)$ and $T \nvdash P(\xi)$, but **M** models $P(\xi)$, where **M** makes T arithmetically consistent.

2. There is a denumerable set of expressions for functions $\xi_m(x) \in B$, $m \in \omega$, such that there is no general decision procedure to ascer-

tain, for an arbitrary m, whether P (ξ_m) or $\neg$ P (ξ_m) is provable in T.

3. Given the set $K = \{m: T \vdash P(\hat{m})\}$, and given an arbitrary total recursive function $g:\omega \rightarrow \omega$, there is an infinite number of values for m so that $C_T(P(\hat{m})) > g(\| P(\hat{m}) \|)$. □

Here $\hat{m}$ recursively codes the set ξ_m of expressions in L_T.

That result was the first general incompleteness theorem obtained by the authors [4]; it can be derived from Rice's theorem [25] [28] in computer science, which is an equally general result, but the proof we originally gave for Proposition A.8 is weaker than Rice's theorem, since it only leads to unsolvable problems of Turing-degree not higher than **0'**.

Problems equivalent to Fermat's Conjecture

A related result is the equivalence between proving famous arithmetic conjectures and the provability of a given nontrivial property P within our formal system [13]. We need:

Proposition A.9 If T is arithmetically consistent then we can explicitly obtain in it a polynomial $p(<x,y,z,m>,v_1,\ldots,v_k)$ over **Z** such that for all $x,y,z,m \in \omega$, $x,y,z>1$ and $m>2$,

$$x^m + y^m \neq z^m$$

if and only if for all x,y,z,m as above,

$$\forall v_1,\ldots, v_k \in \omega \; p(<x,y,z,m>,v_1,\ldots,v_k)>0. \; \square$$

See for the proof either [18] or (explicitly) [13].

Now let $C(x,y,z,m,v) = \lambda\, p(<x,y,z,m>,v_1,\ldots)$, as after Prop. A.3.

Proposition A.10 Given our arithmetically consistent theory T, we can explicitly and algorithmically construct within L_T the formal expression for a function $\phi(x, y, z, m)$ with values in the set {0, 1} such that:

1. $\forall x, y, z, m \in \omega_0 \; \phi(x, y, z, m) = 0$ if and only if Fermat's Conjecture is true.
2. $\exists x, y, z, m \in \omega_0 \; \phi(x, y, z, m) = 1$ if and only if x, y, z, m is a counterexample for Fermat's Conjecture.

Moreover, ϕ (x,y,z,m) can be constructed entirely within the language of elementary real analysis.

Proof: We write the expression

$K(x, y, z, m) = \int_{-\infty}^{+\infty}(C(x, y, z, m, u)e^{-u2} / (1+C((x,y,z,m,u)) du$,
and then put
$\phi(x, y, z, m) = \sigma(K(x,y,z,m) / (1+K(x,y,z,m)))$.
Here $\sigma(\pm x) = \pm 1$ and $\sigma(0) = 0$ is the sign function. □

We can go beyond that and obtain a constant function β such that $\beta = 0$ if and only if Fermat's Conjecture is true, and $\beta = 1$ if and only if Fermat's Conjecture is false, within an arithmetically consistent theory T:

Proposition A.11 Given our arithmetically consistent theory T, we can explicitly and algorithmically construct within L_T the formal expression for a constant function β which is either equal to 0 or 1 such that:

1. $\beta = 0$ if and only if Fermat's Conjecture is true.
2. $\beta = 1$ if and only if Fermat's Conjecture is false.

Moreover, β can be constructed entirely within the language of elementary real analysis.

Proof: Notice that, when extended to the reals in $\mathbf{R}^4$, $\phi(x,y,z,m) \geq 0$. We therefore write:

$$L = \int_{R4} (\phi(x,y,z,m)\exp[-(x^2 + y^2 + z^2 + m^2)] / (1 + \phi(x,y,z,m))) \, dxdydzdm.$$

(Obviously $\exp x = e^x$.) Then $\beta = \sigma(L)$. □

Expressions for functions in higher degrees

Corollary A.12 If T is arithmetically consistent then we can explicitly and algorithmically construct in L_T an expression for the characteristic function of a subset of ω of degree $\mathbf{0}''$.

Proof: We simply use Theorem 9-II in [28] (p. 132). Details are given in [9]. Let $A \subset \omega$ be a fixed infinite subset of the integers; it is our oracle set.

Definition A.13 The **jump of** A is noted A'; $A' = \{x:\phi_x^A(x)\downarrow\}$, where ϕ_x^A is the A-partial recursive algorithm of index x. □

An oracle Turing machine ϕ_x^A with oracle A can be visualized as a two-tape machine where tape 1 is the usual computation tape, while tape 2 contains a listing of A. When the machine enters the oracle state s_0, it searches tape 2 for an answer to a question of the form "does $w \in A$?" Only finitely many such questions are asked during a converging computation; we can separate the positive and negative answers into two disjoint finite sets $D_u(A)$ and $D^*_v(A)$ with (respectively) the positive and negative answers for those questions; notice that $D_u \subset A$, while $D^*_v \subset \omega - A$. We can view those sets as ordered k- and k^*-ples; u and v are recursive codings for them [28]. The $D_u(A)$ and $D^*_v(A)$ sets can be coded as follows: only finitely many elements of A are queried during an actual converging computation with input y; if k' is the highest integer queried during one such computation, and if $d_A \subset c_A$ is an initial segment of the characteristic function c_A, we take as a standby for D and D^* the initial segment d_A where the length $l(d_A) = k'+1$. $d_{y,A}$ denotes the initial segment of A determined by the (converging) input y.

The oracle machine is equivalent to an ordinary two-tape Turing machine that takes as input $<y, d_{y,A}>$; y is written on tape 1 while $d_{y,A}$ is written on tape 2. When this new machine enters state s_0 it proceeds as the oracle machine. (For an ordinary computation, no converging computation enters s_0, and $d_{y,A}$ is empty.) It is a well-known fact that there is a 1-1 recursive function ρ that maps indices for oracle machines into indices for Turing machines. Therefore, $\phi^A_x(y) = \phi_{\rho(x)}(<y, d_{y,A}>)$.

Now let us note the universal polynomial $p(n,q,x_1,\ldots,x_n)$. We can define the jump of A as follows:

$$A' = \{\rho(z): \exists x_1,\ldots,x_n \in \omega\ p(\rho(z), <z, d_{z,A}>, x_1,\ldots,x_n) = 0\}.$$

With the help of the λ map defined after Proposition A.3, we can now form a function modelled after the ϕ function in Proposition A.5; it is the desired characteristic function:

$$c_{0'}(x) = \phi(\rho(x), <x, d_{x,0'}>).$$

(Actually we have proved more; we have obtained

$$c_{A'}(x) = \phi(\rho(x), <x, d_{x,A}>),$$

with reference to an arbitrary $A \subset \omega$.)

We write $\phi^{(2)}(x) = c_{0''}(x)$. □

We recall [28]:

Definition A.14 The complete Turing degrees **0**, **0'**, **0''**,... $\mathbf{0}^{(p)}$,..., $p < \omega$, are Turing equivalence classes generated by the sets 0, 0', 0'', ...,

$\mathbf{0}^{(p)}, \ldots$ □

Now let $\mathbf{0}^{(n)}$ be the n-th complete Turing degree in the arithmetical hierarchy. Let $\tau(n,q) = m$ be the pairing function in recursive function theory [28]. For $\phi(m) = \phi(\tau(n,q))$, we have:

Corollary A.15 (Complete Degrees.) If T is arithmetically consistent, for all $p \in \omega$ the expressions $\phi^p(m)$ explicitly constructed below represent characteristic functions in the complete degrees $\mathbf{0}^{(p)}$.

Proof: From Proposition A.12,

$$\phi^{(0)} = c_0(m) = 0,$$
$$\phi^{(1)}(m) = c_{0'}(m) = \phi(m),$$
$$\phi^{(n)}(m) = c_{0(n)}(m),$$

for c_A as in Proposition A.12. □

Incompleteness theorems

We now state and prove several incompleteness results about **N** and its extension T; they will be needed when we consider our main examples. We recall that the truncated difference operation "$\dot{-}$,"

$$x \dot{-} y = \begin{cases} x - y, & x - y \geq 0, \\ 0, & x - y < 0, \end{cases}$$

is a primitive recursive operation on ω.

Proposition A.16 If T is arithmetically consistent, then we can algorithmically construct a polynomial expression $p(x_1, \ldots, x_n)$ over **Z** such that $\mathbf{M} \models \forall x_1, \ldots, x_n \in \omega\ p(x_1, \ldots, x_n) > 0$, but

$$T \nvdash \forall x_1, \ldots, x_n \in \omega\ p(x_1, \ldots, x_n) > 0$$

and

$$T \nvdash \exists x_1, \ldots, x_n \in \omega\ p(x_1, \ldots, x_n) = 0.$$

Proof: Let $\xi \in L_T$ be an undecidable sentence obtained for T with the help of Gödel's diagonalization; let n_ξ be its Gödel number and let m_T be the Gödel coding of proof techniques in T (of the Turing machine that enumerates all the theorems of T). For a universal polynomial $p(m, q, x_1, \ldots, x_n)$ we have:

$$p(x_1, \ldots, x_n) = (p(m_T, n_\xi, x_1, \ldots, x_n))^2.$$ □

Corollary A.17 If **N** is arithmetically consistent then we can find within it a polynomial p as in Proposition A.16. □

Proposition A.18 If **N** is arithmetically consistent and if P is non-trivial then there is a $\zeta \in L_N$ such that $N \models P(\zeta)$ while **N** $\nvdash P(\zeta)$ and **N** $\nvdash \neg P(\zeta)$.

Proof: Put $\zeta = \xi + q(x_1, \ldots, x_n)\upsilon$, for $q = 1 \stackrel{\bullet}{-} (p+1)$, p as in Proposition A.16. □

Remark A.19 Therefore every nontrivial arithmetical property P in theories from arithmetic upwards turns out to be undecidable. □

Definition A.20 $\xi, \zeta \in L_T$ are **demonstrably equivalent** if and only if $T \vdash \xi \leftrightarrow \zeta$. □

Definition A.21 $\xi \in L_T$ is **arithmetically expressible** if and only if there is an arithmetic sentence ζ such that $T \vdash \xi \leftrightarrow \zeta$. □

Proposition A.22 If T is arithmetically consistent, then for every $m \in \omega$ there is a sentence ξ such that **M** $\models \xi$ while for no $k \leq n$ there is a Σ_k sentence in **N** demonstrably equivalent to ξ.

Proof: The usual proof for **N** is given in Rogers [28], p. 319. First notice that

$$0^{(m+1)} = \{x : \phi_x^{0(m)}(x)\}$$

is recursively enumerable but not recursive in $0^{(m)}$. Therefore, $\overline{0^{(m+1)}}$ isn't recursively enumerable in $0^{(m)}$, but contains a proper $0^{(m)}$-recursively enumerable set. Let's take a closer look at those sets.

We first need a lemma: form the theory $T^{(m+1)}$ whose axioms are those for T plus a denumerably infinite set of statements of the form "$n_0 \in 0^{(n)}$," "$n_1 \in 0^{(m)}$," … , that describe $0^{(m)}$. Then,

Lemma A.23 If $T^{(n+1)}$ is arithmetically consistent, then $\phi_x^{0(m)}(x)\downarrow$ if and only if $T^{(m+1)} \vdash \exists x_1, \ldots, x_n \in \omega\ p(\rho(z), < z, d_{y,0(m)} >, x_1, \ldots, x_n) = 0$.

Proof: Similar to the proof in the non-relativized case; see [25], p. 126 ff. □

Therefore we have that the oracle machines $\phi_x^{0(m)}(x)\downarrow$ if and only if

$T^{(m+1)} \vdash \exists x_1, \ldots, x_n \in \omega\, p(\rho(z), < z, d_{y,0(m)} >, x_1, \ldots, x_n) = 0.$

However, since $\overline{0^{(m+1)}}$ isn't recursively enumerable in $0^{(m)}$ then there will be an index $m_0(0^{(m)}) = < \rho(z), < z, d_{y,0(m)} >>$ such that

$M \models \forall x_1, \ldots, x_n [p(m_0, x_1, \ldots, x_n)]^2 > 0,$

while it cannot be proved neither disproved within $T^{(m+1)}$. It is therefore demonstrably equivalent to a Π_{m+1} assertion. □

Now let $q(m_0(0^{(m)}), x_1, \ldots) = p(m_0(0^{(m)}), x_1, \ldots))^2$ be as in Proposition A.22.
Then:

Corollary A.24 If T is arithmetically consistent, then for:

$\beta^{(m+1)} = \sigma(G(m_0(0^{(n)})),$

$G(m_0(0^{(n)})) = \int_{-\infty}^{+\infty}(C(m_0(0^{(n)}),x)e^{-x^2} / (1+C(m_0(0^{(n)}),x))\, dx,$

$C(m_0(0^{(n)}),x) = \lambda q(m_0(0^{(n)}),x_1,\ldots, x_r),$

$M \models \beta^{(m+1)} = 0$ but for all $n \leq m+1$, $T^{(n)} \nvdash \beta^{(m+1)} = 0$ and $T^{(n)} \nvdash \neg(\beta^{(m+1)} = 0)$. □

Corollary A.25 If T is arithmetically consistent and if L_T contains expressions for the $\phi^{(m)}$ functions as given in Proposition A.15, then for any nontrivial predicate P in **N** there is a $\zeta \in L_T$ such that the assertion $P(\zeta)$ is T-demonstrably equivalent to and T-arithmetically expressible as a Π_{m+1} assertion, but not as any assertion with a lower rank in the arithmetic hierarchy.

Proof: As in the proof of Proposition A.18, we write:

$\zeta = \xi + [1 \dot{-} (p(m_0(0^m), x_1, \ldots, x_n) + 1)]\upsilon,$

where p(...) is as in Proposition A.22. □

Corollary A.26 If T is arithmetically consistent then, for any nontrivial property P there is a $\zeta \in L_T$ such that the assertion $P(\zeta)$ is arithmetically expressible, $M \models P(\zeta)$ but only demonstrably equivalent to a Π_{n+1} assertion and not to a lower one in the hierarchy.

Proof: Put

$\zeta = \xi + \beta^{(m+1)}\upsilon$, where one uses Corollary A.24. □

Undecidable sentences outside arithmetic

We recall:

Definition A.27

$0^{(\omega)} = \{ <x,y> : x \in 0^{(y)}\}$,

for x, y $\in \omega$. □

Then:

Definition A.28

$\phi^{(\omega)}(m) = c_{0(\omega)}(m)$,

where $c_{0(\omega)}(m)$ is obtained as in Proposition A.12. □

Still,

Definition A.29

$0^{(\omega+1)} = (0^{(\omega)})'$. □

Corollary A.30 $\mathbf{0}^{(\omega+1)}$ is the degree of $0^{(\omega+1)}$. □

Corollary A.31 $\phi^{(\omega+1)}(m)$ is the characteristic function of a nonarithmetic subset of ω of degree $\mathbf{0}^{(\omega+1)}$. □

Corollary A.32 If T is arithmetically consistent, then for:

$\beta^{(\omega+1)} = \sigma\, (G\, (m_0(0^{(\omega)})),$

$G(m_0(0^{(\omega)})) = \int_{-\infty}^{+\infty}(C(m_0(0^{(\omega)}),x)e^{-x2} / (1+C(m_0(0^{(\omega)}),x))dx,$

$C(m_0(0^{(\omega)}),x) = \lambda q(m_0(0^{(\omega)}), x_1, \ldots , x_r),$

$M \models \beta^{(\omega+1)} = 0$ but $T \nvdash \beta^{(\omega+1)} = 0$ and $T \nvdash \neg\, (\beta^{(\omega+1)} = 0)$. □

Proposition A.33 **(Nonarithmetic incompleteness.)** If T is arithmetically consistent then given any nontrivial property P:

1. There is a family of expressions $\zeta_m \in L_T$ such that there is no general algorithm to check, for every m $\in \omega$, whether or not $P(\zeta_m)$ in T.
2. There is an expression $\zeta \in L_T$ such that $\mathbf{M} \models P(\zeta)$ while $T \nvdash P(\zeta)$ and $T \nvdash \neg\, P(\zeta)$.
3. Neither ζ_m nor ζ are arithmetically expressible.

Proof: We take:

1. $\zeta_m = x\phi^{(\omega+1)}(m) + (1-\phi^{(\omega+1)}(m))y$.
2. $\zeta = x + y\beta^{(\omega+1)}$.
3. Neither $\phi^{(\omega+1)}(m)$ nor $\beta^{(\omega+1)}$ are arithmetically expressible. □

Remark A.34 We have thus produced out of every nontrivial predicate in T intractable problems that cannot be reduced to arithmetic problems. Actually there are infinitely many such problems for every ordinal α, as we ascend the set of infinite ordinals in T. Also, the general nonarithmetic undecidable statement $P(\zeta)$ has been obtained without the help of any kind of forcing construction. □

Finally, let $Q(x, a_1, a_2, \ldots, a_n)$ be a Suppes predicate on the fixed parameters $a_1, \ldots$. Suppose given an enumeration of the predicates P_k in T. Again we suppose that:

1. For $\xi_i \in L_T$, $T \vdash Q(\xi_i)$
2. For $\xi_i, \xi_j, i \neq j, \in L_T$, $T \vdash P_k(\xi_i) \wedge \neg P_k(\xi_j)$.
3. Out of that we list all nontrivial predicates P'_k that apply to Q-defined objects.

Proposition A.35 If T is arithmetically consistent then:

. **Undecidability.** There is a countable family ζ_m of expressions for Q-objects in T such that there is no general algorithm to decide, for any nontrivial Q-property P_k in T whether that expression satisfies (or doesn't satisfy) P_k.

. **Incompleteness.** There is a Q-object all whose nontrivial Q-properties cannot be proved within T. □

Those are our faceless objects. They seem to lie outside the reach of forcing techniques, as they are defined with the help of nonstandard models. Anyway, incompleteness of the nastiest kind is to be expected everywhere in the axiomatized sciences. For density theorems related to those intractable problems see [13].

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