Philosophica 87 (2012) pp. 117-152

ALGEBRAIC SYMBOLISM IN THE FIRST ALGEBRAIC WORKS IN THE IBERIAN PENINSULA

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ABSTRACT

In this paper, we analyze the symbolic language used in some algebraic works of the second half of 16th century in the Iberian Peninsula. Examples show that in some cases the use of specific symbols was due to the constrictions of typography. In other cases symbolic language was used only as a simplification to contribute to a better understanding of rhetoric reasoning, and finally it was used as a part of symbolic reasoning, thereby moving forward from operations with numbers to operations with new objects of algebra. The works we refer to in this paper were written by the Iberian authors, Marco Aurel (fl. 1552), Juan Pérez de Moya (c. 1513–c.1597), Antic Roca (c. 1530–1580), Pedro Núñez (1502–1578) and Diego Pérez de Mesa (1563– c. 1633). We focus both on the symbolism for the unknowns and on the symbols to indicate the operations, as well as on the way these authors solved the systems of equations. This analysis will contribute to a better understanding of the status of algebra in works from the Iberian Peninsula in the 16th century.

1. Context

In 16th century Europe mathematics underwent deep changes whose diffusion was facilitated by the invention of printing in the previous century, which completely changed the way that culture was transmitted. One of the main changes was the progressive development of algebra from practical arithmetic.

Mathematics is the least studied discipline of the 16th century in Spain. In part, this is due to what is known as the controversy of Spanish science, which arose after a series of unfounded appreciations from authors like Picatoste,[1] Fernández Vallín[2] and Menéndez Pelayo.[3]

[1] Felipe Picatoste (Madrid 1834- Madrid 1892) was the author of *Apuntes para una biblioteca científica española del siglo XVI: estudios biográficos y bibliográficos de ciencias exactas físicas y naturales y sus inmediatas aplicaciones en dicho siglo* (1891), which consists of the biographies of Spanish scientists who were important in their own fields during the 16th century. He wrote this book as a reaction to Echegaray's address on his election to the Academy of Sciences, in which he reviled the Spanish scientific tradition.

[2] Acisclo Fernández Vallín (Gijón 1825 – Madrid 1896) was the author of *Cultura científica de España en el siglo XVI.*

[3] Marcelino Menéndez Pelayo (Santander 1856 – Santander 1812), best known as a literary critic, wrote *The Spanish Science* (1876) in which he laid claim to the existence of a Spanish scientific tradition.

Rey Pastor[4] analyzed some of these texts and demonstrated that some of this praise was without justification. Rey Pastor's research was aimed at, on the one hand, finding reasons that would make Spaniards proud of their country or otherwise sadden them, and on the other hand, seeking a great national figure. For this reason, Pastor failed to take into account the processes of diffusion and transmission of knowledge; neither did he see scientific activity as part of a social reality. His work "Spanish Mathematicians of the 16th century" has for many years been the only source consulted on this topic, and regard for him as an authority has been a barrier to knowledge for the few research projects undertaken into this subject. It is also necessary to take into account the fact that Spain's communication with the rest of Europe virtually ceased from 1557, when groups of protesters were arrested in Seville and Valladolid. One year later, King Philip II presided over the first of a series of autos-da-fe that culminated in the burning of the Spanish Protestants. This ideological repression constituted a strict control of intellectual activity by both the monarchy and the Inquisition, which at first was confined to theology but soon spread to other fields. These circumstances made the availability of European reference works more difficult in Spain.

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[4] Julio Rey Pastor (Logroño 1888 – Buenos Aires 1962) was a mathematician who made some incursions into the history of mathematics. In his inaugural speech for the 1913 academic year at Oviedo University, he followed Echegaray's line in wresting importance from Spanish mathematicians.

The best known and, probably, the most influential Renaissance algebra, was the *Summa de Arithmetica, Geometria, Proportioni and Proportionalità* of Luca Pacioli (1445–1514). It was written in vernacular and was a compilation of unpublished works that the author had composed earlier, as well as of general mathematical knowledge at the time. It was, together with Euclid's *Elements,* a reference work for the Iberian authors in 16th century.

2. Marco Aurel

The first known book to be printed in the Iberian Peninsula that could be regarded as an algebraic treatise was *Despertador de ingenios. Libro Primero de Arithmetica Algebratica,*[5] which was published in Valencia in 1552. Its author, Marco Aurel, was born in Germany and settled in Valencia, where he taught practical mathematics.

This book consists of 24 chapters. In the six first chapters Aurel expounds the properties of whole numbers; fractions and proportions; the rule of three; currency exchanges and progressions. From the 7th to

[5] This is the first known algebraic treatise to have appeared in print, but not the first treatise in the Iberian Peninsula containing algebra, as Docampo has shown (Docampo, 2008).

the 12th chapter, he deals with square roots and cube roots, binomials[6] and residuals. In the 13th chapter he states the characters, or notation, that he will subsequently use. In the 14th and 15th chapters, from the 17th to the 21st, and in the 23rd and 24th chapters, he deals with the different types of equalities and with the rules to solve them. [7] In the 16th chapter, Aurel addresses the rule of quantity and in the 22nd the roots of binomials and residuals.[8]

Aurel refers to *algebra* as a rule that was commonly known as the *regla de la cosa*[9], although he says that according to Guillelmo de Lunis its correct name was *Algebra & Almucabala*.[10] In fact, Aurel was to assign to

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[6] In most of the first algebra treatises that appeared during the Renaissance, we find a chapter about the classification of binomials and apotomes, which are the expressions consisting of an addition (in the case of binomials) or a subtraction (in the case of apotomes) of a rational number and a square root or of two square roots.

[7] Aurel and other authors called "equalitites" [ygualaciones] what we know call nowadays equations

[8] The *rule of quantity* or *the rule of the second quantity* are the expressions used in the first treatises on algebra to refer to a procedure of solving problems in which more than one unknown was involved.

[9] Rule of the Thing.

[10] Aurel refers to Guillelmo de Lunis as being the first to translate Al-Khwarizmi's *Algebra* from Arabic into Italian (Aurel, 1552, 68v), although it is now considered by some authors that the translation of Lunis was into Latin. More information in Heeffer (2008, 91-92).

algebra the same names as Pacioli : *la regla de la cosa*, or *Arte Mayor*, or *Algebra e Almucabala*, which Aurel translates as *restauratio e oppositio*[11] as Pacioli had done. He states that the Rule of the Thing is based on a continuous proportion of *cossic numbers* such as squared or cubic numbers.

What exactly does Marco Aurel mean by *algebra?* Is it a process for obtaining an equation from the wording of a problem? Is it a rule for solving every type of equation? Or is it the whole process? We consider the two approaches adopted by Aurel for finding a square root of a binomial:

Chapter XI. Roots of binomials	Chapter XXII. Roots of binomials by "the thing"
I want to know the √ of $27+\sqrt{200}$. The two powers[12] are 729 and 200, whose difference is 529, the √ of which is 23; these together with the greater part of the binomial will be 50, half of which is 25, which is the "power" of the greater part; when these are	$27+\sqrt{200}$ is first binomial. Divide 27 into two parts, which when multiplied together will give 50, which is the fourth part of the "power" of $\sqrt{200}$. Assuming that one part is x, the other will be 27Q-1x:[15] by

[11] Aurel, (1552, 68v).

[12] Here Aurel says "powers" [potencias] when he refers to a "squares".

[15] The symbols used by Aurel can be found on the following page.

subtracted from the greater part we obtain 2, which is the "power" of the lesser part of the square root. If we multiply one "power" by the other, we obtain 50, that is, the fourth part of the "power" of the lesser part of the binomial. And the √ of each "power" are $\sqrt{25}$ and $\sqrt{2}$, which joined together will be $5+\sqrt{2}$. You will then say that this fourth[13] binomial is the √ of the $27+\sqrt{200}$, which is a first binomial.[14]	multiplying one of these parts by the other, we obtain 27*x*-1*z*, equal to 50Q, which means 27*x* equal to 1z+50Q. Follow with the equality and the value of x will be 25Q. This is one part, and the other will be 2, which when multiplied by the other will give 50. Add the √ of each part together and you will obtain $\sqrt{25}+\sqrt{2}$, that is, $5+\sqrt{2}$, which is the √ of the binomial.[16]

[13] In book X of Euclid's *Elements* we find a classification of binomials into 6 types, and Aurel refers to one of these types.

[14] "Quiero saber qual es la √ de $27+\sqrt{200}$. Las dos potencias son 729, y 200: cuya differencia es 529: √ de la qual es 23: los quales junta con la parte mayor del binomino, y serán 50, cuya metad, es 25, que es la potecia de la parte mayor: los quales resta, o quita de la parte mayor, del binomino, y quedaran 2, que es la potencia de la parte menor del binomino. Y la √ de cada una destas dos potencias, es $\sqrt{25}$, y $\sqrt{2}$: juntados seran $5+\sqrt{2}$. Este binomino 4º diras que es √ de $27+\sqrt{200}$ "(Aurel, 1552, 63r).

[16] "Binomino pº es, $27+\sqrt{200}$, Parte 27 en 2 tales partes, que multiplicando la una con la otra, vengan 50, q es la 4ª parte de la potencia de $\sqrt{200}$. Pongo q la

The first approach is an example to illustrate the method of extracting the square root of a first binomial[17] using the instructions from the tenth book of Euclid's *Elements*[18] and the second is an example from 22th chapter where Aurel deals with extracting the roots of the binomials using the Rule of the Thing.

We can see that the procedure is the same; however, in the 11th chapter Aurel follows a sequence of instructions to obtain the square root of a binomial, while in the 22th chapter, where the author says that he finds the same root by "the thing", he translates one part of the instructions into algebraic language. Therefore he considers the "rule of thing" not only as a rule for solving the equations, as Pacioli seems to regard it, but as the whole process.

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una parte sea 1x, la otra sera 27Q-1x: multiplicado la una con la otra, vienen 27x-1z, ygual a 50Q. Yguala, y vernan 27 x, ygual a 1z+50Q. Sigue la ygualacion, y verna la x a valer 25Q. Tanto diras que es la una parte: la otra será, 2, multiplicada la una con la otra, vienen 50. Agora junta la √ de cada una de las dos partes juntas, y vernan $\sqrt{25}+\sqrt{2}$, digo, $5+\sqrt{2}$. Tanto dira que es la √ del suso dicho binomino" (Aurel, 1552, 133r).

[17] See footnote 6.

[18] Aurel refers to propositions 48, 49, 50, 51, 52 and 53 from the tenth book of *Elements* (Aurel; 1552, 62r–65r). The version he used cannot be deduced from his text. These propositions correspond to Propositions 54, 55, 56, 57, 58 and 59 in Heath's version (Heath, 1956, 116-132).

The characters, or notation, which the author is going to use are stated in the 13th chapter, and he says that these characters are chosen for shortness, as a simplification of the language. Thus, the intention of the author is to make the explanations clearer. In fact, the brevity of the notation will become important in improving algebraic reasoning although the simplifications themselves do not constitute an important step in the algebraic thinking.

He expresses as follows:

> ...I want to put 10 characters in continued proportion and call each of them by their name... the first one is called dragma or number; the second one root or thing...
> One writes the characters because they are short and to avoid writing them with all the letters ...everyone can put the characters he wants...I put the following ones:[19]

[19] "...quiero poner 10 caracteres en una continua proporción, y nombrar a cada uno por si, por su propio nombre que le conviene... el primero se llama dragma, o numero; el segundo rayz, o cosa... Ponense los caracteres, porque son breves, y por evitar la prolixidad de escribir tales nombres a la larga...cada uno puede poner los que a el plazeran...Yo al presente pongo los siguientes" (Aurel, 1552, 69r).

℈ Dragma, o Numero, aſſi ℊ. Radix, o coſa aſſi, ℞. Cenſo aſſi, ℥. Cubo aſſi ce. Cenſo de cenſo aſſi ℥℥. Surſolidum, o primo relato, aſſi ß. Cenſo y cubo aſſi, ℥ce. Biſſurſolidum aſſi, bß. Cenſo cenſo de cenſo aſſi, ℥℥℥. Cubo de cubo aſſi, cce.

(Marco Aurel, 1552, 69r)

This notation clearly shows a German influence that is probably due to the author's own origin. In particular, the influence of Rudolff's work can be seen in the wording of several problems, in the way they are solved, and in the notation, which Rudolff gave as follows:

ℊ dragma oder numerus
℞ radix
℥ zenſus
ce cubus
℥℥ zenſdezens
ß surſolidum
℥ce zenſicubus
bß biſſurſolidum
℥℥℥ zenſzenſdezens
cce cubus de cubo

(Rudolff, 1525, 174)

To indicate addition and subtraction, Aurel used the signs + and –. To indicate the equality, he wrote the word "ygual" and put a symbol [Q] with the independent term.[20]

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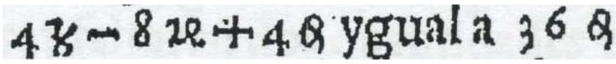
[20] Equal.



(Aurel, 1552, 77r)

In modern notation,

$$4x^2 - 8x + 4 = 36$$

The Iberian authors classified equations [equalitites], into different types, the classification varying according to each author. Aurel considered eight types: the first four types were called simple equalities and can be expressed with only two quantities, while the other four were called compound equalities and three quantities are needed to express them. The authors gave rules, rhetorically expressed, to solve every type of equation. These rules contributed to improving algebraic thinking, although we consider that these rules themselves did not constitute a relevant step in the process of algebraization of mathematics.

However, it is our belief that a real step forward in the algebraization of mathematics was the use of a second unknown, which contributed to the evolution of the idea of an equation, taking it from a tool for problem solving to the understanding of an equation as a new object of algebra[21]. According to Franci, the second unknown appeared for the

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[21] It is not easy to follow the thread of the second unknown throughout the different texts when trying to find some influences among the authors. A deep study on the early occurrences of the second unknown in European texts can be found in (Heeffer, 2010) and (Romero, 2011).

first time in Western culture around 1373 in Antonio de Mazzinghi's[22] *Trattato di Fioretti.*

Aurel deals with the second quantity in the 16th chapter of his algebra, after the chapter devoted to solving the first type of simple equalities and before the chapter devoted to solving the second type of simple equalities. The inclusion of this chapter among others devoted to simple equalities shows that this author considers the "rule of quantity", or cases with a second unknown, as particular cases of the first type of simple equalities.

The heading to 16th chapter of Aurel's work reads as follows:

> Trata de la regla de la quantidad, con algunas reglas, y demandas que por ella se hacen, que por otro nombre se puede llamar, regla de la segunda cosa.[23]

The introduction in which Aurel expresses the need of a new "position", which he names *q*, when only one "x" is not enough, is similar to that by Pacioli. Aurel solves eight problems, which are solved by a method similar to Rudolff's in his *Coss*. It consists in putting the second unknown in terms of the first one, the same for the third unknown and so on. *Co*. being the first unknown, he puts *q* for the

[22] Franci (1988, 246).

[23] Aurel (1552, 108r). It deals with the rule of quantity as well as with some rules and requirements that can be done with it, and to which one can refer as a rule of second quantity.

second, and when this is expressed in terms of the first unknown he also puts *q* for the third and so on. This method, which assigns the same name to the second and the following unknowns, does not allow these authors to address a whole system of equations; they are obliged to work with every unknown and every equation, and are unable to work with all the equations at the same time.

Let us illustrate the procedure with the solution to the second problem, which has the same wording as problem 191[24] in Rudolff's *Coss*:

> Three friends have a certain amount of money and they want to buy a horse whose price is 34 ducats. The first friend tells the others to give him the half of money they have and with the money he has, he will buy the horse. The second tells the others that if they give him a third of the money they have, he could also buy the horse. Finally, the third friend wants the others to give him a quarter of the money they have, and by adding his own money he will buy the horse (Aurel, 1552, 108 v).

When we translate the wording of the problem into current algebraic language, we obtain:

[24] See Rudolff (1525, 221).

$$\begin{cases} x+\frac{1}{2}y+\frac{1}{2}z=34 \\ \frac{1}{3}x+y+\frac{1}{3}z=34 \\ \frac{1}{4}x+\frac{1}{4}y+z=34 \end{cases}$$

Aurel solves the problem in the following way: let us assume the first of the companions has *x* ducats. So, he needs 34*Q*-1*x* to be able to buy the horse, a quantity that has to be equal to half of the other two.[25] Therefore, the other two have 68*Q* - 2*x* ducats and altogether the three of them have 68*Q* - 1*x*. If the second one has *q* ducats, then the first and the third one together have 68*Q* - 1*x* - 1*q* ducats. A third of this quantity is

$$22\frac{2}{3}Q-\frac{1}{3}x-\frac{1}{3}q\ ,$$

to which we have to add the quantity that the second one has, so that we will get:[26]

$$22\frac{2}{3}Q-\frac{1}{3}x+\frac{2}{3}q\ .$$

Then he makes this quantity equal to 34 ducats, which is the price of the horse, and with this expression the quantity of the second friend can be expressed in terms of the quantity of the first.

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[25] Aurel is working with the expression that corresponds to a first equation in the system above.

[26] Now Aurel is working with the second equation of “our” system.

He then assumes that the third friend has q ducats; that is, he has used q to set the quantity that the second one has in terms of the first one's quantity, and then he uses the same unknown again to indicate the money that the third one has. As before, the first and the second friend together have $68Q - 1x - 1q$ ducats. One quarter of this quantity is:[27]

$$17Q - \frac{1}{4}x - \frac{1}{4}q$$

to which he adds the quantity q that the third one has and obtains

$$17Q - \frac{1}{4}x + \frac{3}{4}q$$

which must be equal to the price of the horse, that is, 34 ducats. From this equality he gets the expression of the third friend's quantity in terms of the first one:

$$\frac{1}{3}x + 22\frac{2}{3}Q \; .$$

When he joins the three quantities together, he obtains:

$$1x + \frac{1}{2}x + 17Q + \frac{1}{3}x + 22\frac{2}{3}Q = 1\frac{5}{6}x + 39\frac{2}{3}Q \; ,$$

which must be equal to $68Q - 1x$, from which he gets $x = 10$, and *22* and *26* for the second and third one's quantities of ducats.

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[27] Aurel is now working with the expression that corresponds to our third equation.

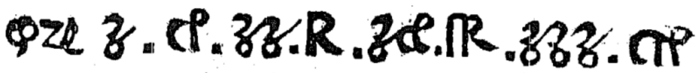
3. Juan Pérez de Moya

Aurel's book is probably one of the main sources for the most popular book written in the Iberian Peninsula during the 16th century, *Arithmetica practica y speculativa* (Salamanca, 1562) by Juan Pérez de Moya (Santisteban del Puerto, ca. 1513 – Granada, ca. 1597), which ran to 30 editions, from the first in 1562 to the last in 1798, being the most popular mathematical text for Spaniards during this period. In 1558, Pérez de Moya wrote the *Compendio de la Regla de la Cosa, o Arte Mayor,* which four years later was to become the 7th book of his *Arithmetica practica y speculativa.*

This *Arithmetica* consists of nine books which deal with the properties of whole numbers, fractions, the rule of three, currency exchanges and also some practical rules of geometry. The last book is very different from the rest. It does not contain rules and it is written as a dialogue between students about the importance of arithmetic, which the author wishes to highlight. The 7th book, which deals with the Rule of the Thing, is divided into fifteen chapters.

The first chapter of the *Compendio de la Regla de la Cosa, o Arte Mayor* and the first chapter of the 7th book of the *Arithmetica practica y speculativa* are brief introductions to the Rule of the Thing. First, the author states that this rule has different names, depending on the author, some of whom call it "rule of algebra", which means restoration, or "almucabala", which means solution, because with this rule infinite questions can be solved regarding arithmetic, geometry or other branches of mathematics[28]. Pérez de Moya adds that other authors call it "the Rule of the Thing" and still others "Arte Mayor". The goal is to find an unknown proportional number. Thus, the author considers the algebra as a rule for solving problems of all branches of mathematics. We find a similar idea in Pedro Núñez work.

In the second chapter, Pérez de Moya gives some characters that he says have been devised for their brevity. These are:



Although he gave the same characters as Aurel, he did not quote him as a source. In the third chapter, Pérez de Moya sets out other characters he intended to use, giving as his reason that there are no others in print[29]. These characters are as follows: *n* for the number; *co.* for the thing; *ce.* for the census; *cce.* for the census of census and so on, all of them belonging to the Italian tradition. He considers four types of simple equalities and three of compound equalities, although in fact he discusses the same cases as Aurel.

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[28] *Diversos nombres tiene esta regla, acerca de varios autores. Unos la llaman regla de algebra, que quiere decir, opposicio, o absolucio: porque por ella se hacen, y absuelven infinitas questiones* (y las que son imposibles nos las demuestra). *Asi de Arithmetica como de Geometria, como de las demás artes* (*que dizen*) *Mathematicas* (Pérez de Moya, 1562, 448).

[29] *Capítulo tercero. En el qual se declaran algunos caracteres que yo uso, por no aver en la stampa otros* (Pérez de Moya, 1562, 452).

He used no special symbol to indicate addition and subtraction, but rather the letters *p* and *m*, abbreviations of the Italian words "più" and "meno" that were used by Pacioli in his *Summa.* To indicate equality he abbreviated the Spanish word "ygual" by putting "yg", and he also put a symbol with the independent term.

18. ço.m. 22. n̄.yg.à.10.co.p. 8. n̄.

(Pérez de Moya, 1562, 563)

In modern notation this would be written:

$$18x - 22 = 10x + 8$$

The 9th article of the 13th chapter of the 7th book of the *Arithmética práctica y speculativa* (1562 edition) is about the rule of the second thing or quantity. He solved only three problems, all of them concerning numbers without any special context, and the approach he adopted was the same as that used by Pacioli and Aurel. However, the article in which Pérez de Moya deals with the "rule of quantity" is not included among the articles referring to the rules of equalities, but rather at the end of the chapter, constituting for the author not a particular case of the seventh types of equalities, as was for Aurel, but rather a special case. In the 1573 *Tratado de Mathematicas,* the author extended his work, particularly in relation to the second quantity. In the third chapter of the seventh book, he gave the same characters as in the *Arithmética* *práctica y speculativa,* and as in this previous work he uses Italian notation. He did not quote Aurel in this work either.

In *Tratado de Mathematicas,* the "rule of quantity" is found in the sixth article of the 61st chapter, where six problems are solved with significant differences from the *Arithmética práctica y speculativa.* The most noteworthy novelty is the assignation of a letter to the third unknown, different from the letter assigned to the second one. He uses the word "quantity" for the second unknown, but instead of naming it "q" he calls it "a". If a third unknown is needed, he names it "b".[30] Pérez de Moya solved these problems rhetorically, by substitution, finding the value of one unknown in one of the equations and substituting in the other. He wrote the equations symbolically among the explanations, but not the whole system of equations.

4. Pedro Núñez

The most remarkable mathematician in the Iberian Peninsula in the XVI century was probably the Portuguese Pedro Núñez (Alcácer do Sal, 1502–Coimbra, 1578), who studied medicine and mathematics and was professor at the University of Salamanca and also in Coimbra.

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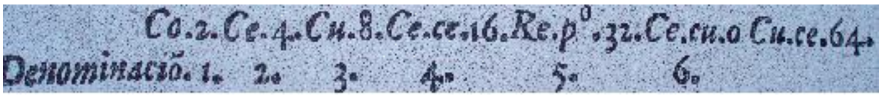
[30] This type of assignation, which gives relevance to the first unknown, is also highlighted in Stifel's comment of *Die Coss* (Stifel, 1553, 186r) by Rudolff, but there is no evidence that Pérez de Moya was inspired by Stifel's comment.

Núñez's book *De algebra en arithmetica y geometria,*[31] which he wrote in Portuguese during the 1530s, was not published until 1567 and appeared in Spanish. It is a very long work, and differs from Aurel's and Pérez de Moya algebras. The interval between when it was written and its publication makes it difficult to analyse because there are many references to works published after the 1530s that could not have been in the original version and which helped to shape the work. In the preface, Núñez referred to the importance of Euclid's *Elements* for the foundations of algebra, and he also referred to the *Summa* by Pacioli as the first printed algebra treatise, known only to a few people in Spain 60 years after its publication, due, according to Núñez, to the obscure way that it was written.

Núñez's algebra is divided into three parts, although in fact it consists of five parts because the second one is divided into three further independent parts. In the first part, Núñez gave the resolution for the first and second degree equations with one unknown. It has six chapters, the first one dealing with the purpose of algebra, its conjugations (the name that Núñez used for equations) and its rules. He stated that the purpose of algebra is to highlight the unknown quantity,

[31] An extensive study on this work can be found in Leitao (2010), Labarthe (2010), Romero (2010), Loget (2010) and Massa (2010).

and that the means he will adopt is the use of the equality[32]. The quantities are: *number*, *thing* and *census*, which refers to as “dignidades” or ‘powers’, and in the first chapter of the second part he assigns names to the first six and explains the way for obtaining the others.


Co.2.Ce.4.Cu.8.Ce.ce.16.Re.p°.32.Ce.cu.o Cu.ce.64.
Denominació. 1. 2. 3. 4. 5. 6.

(Núñez, 1567, 24v)

Núñez considered three types of simple conjugations and three more compound conjugations, each of which has its own rule. He used the symbol *p* to indicate addition and *m* to indicate subtraction. For equality, he writes the whole word ‘yguales’ in Spanish. He put no symbol with the independent term.


20.co. p̃.7.ce. yguales a 30.co.p̃ 4.ce.m.3

(Núñez, 1567, 124r)

In modern notation:

$$20x + 7x^2 = 30x + 4x^2 - 3$$

In the 5th and 6th chapters of the third main part, Núñez solved some problems that nowadays we would solve as systems of equations:

[32] “En esta Arte de Algebra el fin que se pretende, es manifestar la quantidad ignota. El medio de que usamos para alcançar este fin, es igualdad” (Núñez, 1567, 1r).

in the 5th, *De la Practica de las Reglas de Algebra en los casos de Arithmetica,* he solved 110 problems than can be resolved by using systems of equations, but he made no mention of a rule of quantity.

In the 6th chapter, Núñez dealt with the rule of quantity, *About the rule of simple or absolute quantity,* and solved only three problems. He stated that the rule of simple or absolute quantity is different from the others and that it can be used in two ways. The first way is a substitution of the *rule of the thing* to perform the equality with the help of the term *quantity,* while the second is to do it « position » over « position ». With this last expression, Núñez referred to problems whose wordings contain one unknown. The first problem he posed is akin to that from Aurel about three men who want to buy a horse, but without any context. It is also similar to the 26th question in *distinctio nona, tractatus nonus* in *Summa* de Pacioli and it is also the first problem that Cardano solves in Chapter IX of his *Ars Magna.* Indeed, it is also question IIII in Pelletier's first book of algebra[33], with a similar wording to Cardano's referring to a three men that have some amount of money.

Nuñez's wording is as follows:

> We have three numbers, the third number with half of the other two equals 32, the second number with a third of the other two equals 28, and the third number with a

[33] Pelletier (1554, 107).

> quarter of the other two equals 31. We want to know what each of them is (Núñez, 1567, 224v).

To solve this kind of problem, Núñez put *co.* for the first unknown and « quantity » for the second, without using any abbreviation. Using similar reasoning to Aurel, and also rhetorically, he put the « quantity » in terms of the « *co.* ». He worked with the first two equations and also put the third unknown in terms of the first, but without assigning any name to the third unknown.

The third and last problem that Núñez included in the 6th chapter leads to the solving of a quadratic equation. The author says that Cardano solved this problem by *the rule of quantity*, but that it can be solved in an easier way by *the rule of the thing,* by being a little skeptical in the use of this rule, due perhaps to the use of the same letter for the second and third unknowns.

5. Antic Roca

Antic Roca (c.1530–1580) was born in Girona and graduated in Arts and Philosophy in 1555 at the University of Barcelona, where in 1559 was elected a professor in Arts.[34]

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[34] The study of Arts was a preparatory course for the higher faculties of theology and medicine.

The *Arithmetica* (1565 edition, the first appeared in 1564) consists of 268 folios divided into two parts, each of which is made up of four books. On the cover of his *Arithmetica*, the author states that it is a compilation of the other arithmetical works that had been published up to that moment.

The first part of the book is basically a practical arithmetic, while the second part, which deals with mercantile problems, contains the "rule of thing". The fourth book of this second part of the *Arithmetica*, deals with the Arte Mayor. The author considers that the Arte Mayor is based in the knowing of "four admirable operations". The first one is about the squared numbers, the second about the cubic numbers the third about the binomials and the fourth about the Rule of the Thing. Thus, for Roca, the Rule of the Thing is not a synonym of algebra, but a part of it.

In the 13th chapter, "De los Characteres que sirven para continuas proporciones, y de los nombres dellos y significaciones".[35] Roca explained his characters, or notation. He stated that characters can be named in different ways and he also noted that the number of characters described is not essential for the use of the rule. Nevertheless, he claimed that the number of characters can be extended infinitely, like the numbers themselves

[35] On the characters that are useful for the continued proportions and their names and significances.

o.	1.	2	3.	4.	5.	6.	7.	8.	9.
N.	co.	ce.	cu.	cce.	r.	cecu.	rr.	ccce.	ccu.

(Roca, 1564, 258r)

He wrote '*mas.*' [36] to indicate addition and '*me.*' [37] to indicate subtraction and, like Aurel and Pérez de Moya, he put a symbol with the independent term as can be seen in the following illustration:[38]

3. cce. me. 8. cu. mas. 4. ce. mas. 7. co. ma. 5. n.
4. ccc. me. 3. cu. me. 7. ce. mas. 5. co. me. 5. n.

7. cce. me. 11. cu. mas. 3. ce. mas. 12.

(Roca, 1564, 256r)

This operation with polynomials in modern notation would be:

3	x^4	-	8	x^3	+	4	x^2	+	7	x	+	5	n
4	x^4	-	3	x^3	-	7	x^2	+	5	x	-	5	n
7	x^4	-	11	x^3	+	3	x^2	+	12	x			

[36] *Mas* is the whole Spanish word to express the addition.

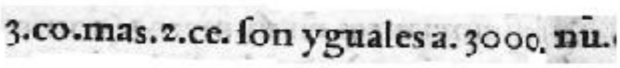
[37] *Me* is an abbreviation of the Spanish word "menos", which expresses subtraction.

[38] In this case there is a mistake in the addition to the "*ce.*"

Before writing this operation, he stated that when the quantities have a different sign, one should subtract the lesser quantity from the greater, although he made no mention of which sign should be used.

As regards the classification of equalities,[39] Roca said that he had decided to follow Aurel and gave 4 simple and 4 compound equalities, all of them corresponding to those used by him, while some expressions literally reproduce those from Aurel.

In the following equation, we can see that for the equality Roca writes the Spanish word "yguales"


3.co.mas.2.ce. ſon yguales a. 3000. nũ.

(Roca, 1564, 263v)

In modern notation,

$$3x + 2x^2 = 3000$$

In the fourth book of his *Arithmetica*, in which he dealt with the Arte Mayor, Roca made no reference to the rule of quantity.

[39] For more information about the treatment of equations in Roca, see Massa (2008, 2012).

6. Diego Pérez de Mesa

Diego[40] Pérez de Mesa[41] was born in Ronda (Málaga) in 1563 and studied Arts[42] and Theology at the University of Salamanca, where he followed the courses taught by Jerónimo Muñoz (València 1520–Salamanca 1592) who occupied the chair of Astronomy and Mathematics.[43]

Pérez de Mesa wrote several mathematic works but did not publish any of them. Manuscript 2294 in the Salamanca University Library is one of these works. It is a double-faced treatise consisting of 100 pages and dated 1598. The first part deals with arithmetic, while the second, which begins on page 60, concerns algebra and is composed of an introduction and 23 chapters. The last six chapters are about the resolution of equations and systems of equations.

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[40] He uses Iacobus as his name when he writes in Latin.

[41] For further information about Pérez de Mesa and his MS 2294, see Romero (2007).

[43] Jerónimo Muñoz was born in València where he studied Arts. After travelling in different countries to complete his education and training, he held the chair of Hebrew at the University of Ancona. He returned to València where he served as professor of Hebrew and mathematics between 1563 and 1578. From 1579 he held the chair of astronomy and mathematics in Salamanca.

In this manuscript Pérez de Mesa states that the algebra is called "regla de la cosa" by some authors, since the "Italians" call "cosa" the "root" or "number".

When referring to the different names by which algebra is known, he quotes "Triputeon" [44] and says that this author refers to algebra as the "ciencia de la cuadratura". Pérez de Mesa quotes "Puteon" later when he refers to the names assigned to a two sides of a "plane number".[45] The two quotations probably refer to the same author, who may be Buteo, the Latinized name of Jean Borrel, who in 1559 changed the name of algebra to *Logistica* or, more specifically, *quadratura,* and to whom Pérez de Mesa referred in another manuscript dated 1593[46] claiming *à Buteone quadratura nuncupati*.[47]

In the twentieth chapter, Pérez de Mesa studied the compound equalities, that is, the equalities that "after removing the superfluous quantities, which are those having the character plus, and adding the tiny quantities, three dimensions still remain".[48] He said that it is not

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[44] Pérez de Mesa (1598, 61).

[45] "...Puteon llama a todo el nº plano contenido y al uno de sus lados continente y al otro lado le dicen ordinariamente cosa...", Pérez de Mesa (1598, 65).

[46] It is part of a MS 19008 of Biblioteca Nacional de España.

[47] That Buteone specified as quadratura.

[48] ...despues de haver quitado lo superfluoques lo que tiene con el caracter mas y añadido lo diminuto quedan 3 dimensiones [Pérez de Mesa, 1598, 92]

important what these dimensions are, although "writers" use the dimensions: nº, £ and q, which can be placed in three ways.

The title of the last chapter on algebra in MS 2294 by Pérez de Mesa is "*de la Regla de la quantidad*" and concerns systems of equations[49]. The way of solving them is different from the approach adopted by other Spanish authors, unlike most of whom, Pérez de Mesa did not take an auxiliary unknown, but rather put « a » for the first unknown, « b » for the second, and so on. Although in his Tratado *de Matematicas,* Pérez de Moya used different names for different unknowns, he nevertheless retained a special name for the first one. Pérez de Mesa, however, considered all the unknowns at the same level. This method could have been taken from Buteo who assigned the letter A to the first unknown and B,C,D,...for the other unknowns and add and subtract equations.

The first exercise that he solved is as follows:

> Two numbers are given, so half of the lower with the higher is 15 and the lower with a third of the higher is 10.[50]

Pérez de Mesa calls the larger number *a* and the smaller *b*. He said that the fractions should first be reduced to whole numbers, and then obtained the system that he set out as follows:

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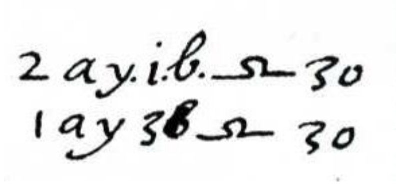
[49] For more information about the equalities in Pérez de Mesa, see Romero (2007, 2008).

[50] Pérez de Mesa (1598, 99).



He solved the system by using the method that nowadays is called "the method of elimination", by multiplying the second equation by 2 and subtracting it from the first equation, thereby obtaining 5b=30, and thus the value 6 for "b". He replaced this value in the first equation to obtain 12 for "a".

Although he solved this problem in a rhetorical way, I would like to point out the symbols he employed. He used the letter "y" for the "plus sign", as he did when operating with polynomial expressions, and put Ω for equality. This is the first time in his manuscript that he used a symbol to indicate equality, which was not introduced until the moment when Pérez de Mesa had to operate with equations, giving to them the status of new algebraic entities.

7. Concluding remarks

Regarding the word *algebra*, for Marco Aurel, it was synonymous with *regla de la cosa* and *Arte Mayor,* and refers to the entire process from the wording of a problem to a solution of the corresponding equation. Pérez de Moya expressed the same idea about *algebra* and remarked that it was a rule for solving problems in all branches of mathematics, as Núñez also remarked. Antic Roca considered that the Arte Mayor was a part of Arithmetic, while algebra was the fourth operation of the Arte Mayor and was the synonym for *regla de la cosa*. Pérez de Mesa also considered that *algebra* was a part of arithmetic.

Regarding symbolism, not all the authors used the same symbols to indicate the unknown and its powers. Aurel took his from the German tradition, while Pérez de Moya, Núñez and Roca took theirs from the Italian tradition. In the case of Pérez de Mesa, the symbolism he used for the unknowns are the first letters of the alphabet. Neither Núñez nor Pérez de Mesa used any symbol to refer to the independent term. However, Aurel, Pérez de Moya and Roca did use different symbols to refer to this term: *Q*, *n* and *nu*, respectively.

In all these authors, the resolution of quadratic equations was rhetorical, and although they stated the symbols that they intend to use, these symbols did not modify the reasoning.

What did indeed represent a big step forward in the process of mathematics algebraization was the use of a second unknown. If we observe its use chronologically in the 16th century Iberian algebraic texts, we can see different stages of algebraic reasoning. Thus, the first stage involved the use of the second unknown by Aurel, who employed the same method as Rudolff. It was Rudolff who referred to the second unknown as the "quantity", as also for the third unknown. Nuñez's method may also be included at this stage, although he was skeptical about the use of the second unknown. The second stage is found in the work by Pérez de Moya, published in 1573 in which different signs are assigned to different unknowns, although the first one retained a special name. We see the most advanced stage in Pérez de Mesa's manuscript, where the author wrote the system of simultaneous equations explicitly, treated the unknowns with equal weight and operated with equations, solving the system by the method that is nowadays called *elimination.* The use of a special symbol, Ω, by Pérez de Mesa to indicate equality, allowed him to consider equations as algebraic objects that can be added or subtracted.

While in most respects the symbols constituted only a simplification of language in order to make explanations easier, in some cases they contributed towards the creation of new algebraic objects, thereby modifying the process of reasoning.

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